Option Pricing Models and Parameter Improvement

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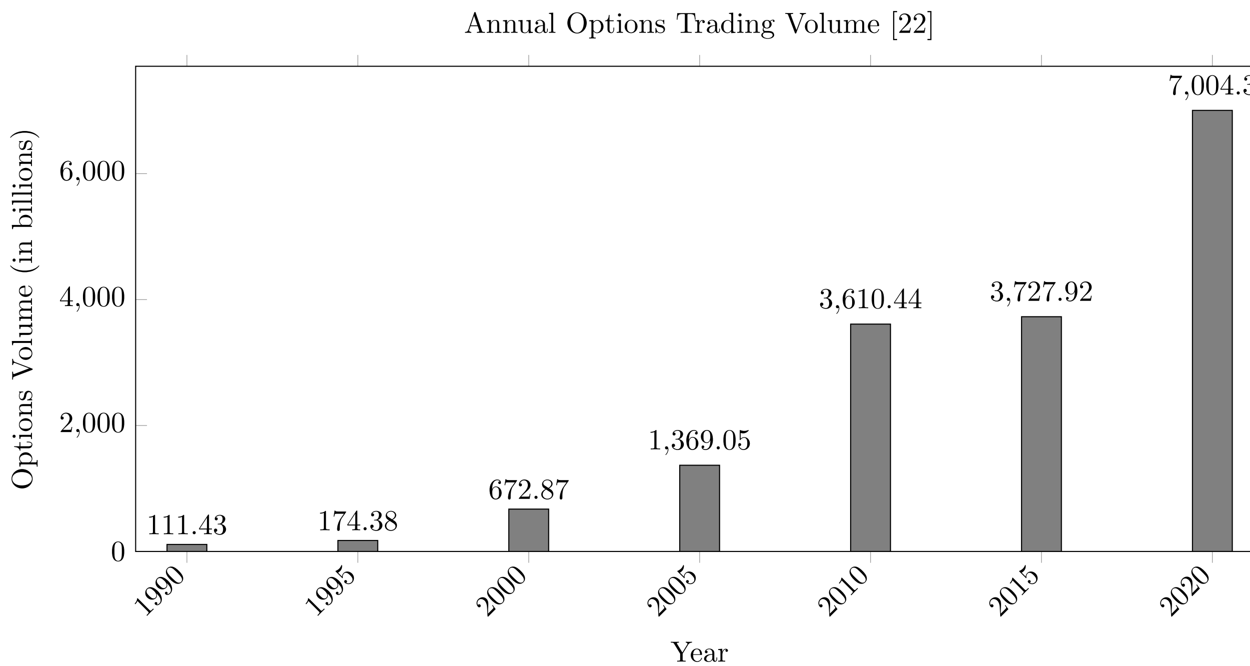
Abstract

The popularity of derivatives trading has consistently risen throughout history. The concept of options has existed since as far back as 10,000 years ago as people had the desire to own assets without physically holding them, or to reserve a sale price for some time in the future.

The most notable spike in recent times was during the 1970s with the formation of the Chicago Board of Options Exchange, followed by the birth of electronic trading in 1992 which revolutionised both the speed and accessibility of trading.



Early derivatives were simple compared to the way they are currently used, now ranging from everyday market speculation to becoming useful in the decentralization of markets via their integration with blockchain technologies. The complexities of such derivatives have been exacerbated by the creation of further higher-order derivatives. A specific type of higher-order derivative is a compound options - an option whose underlying asset is yet another option. [23]

Numerous models have been developed that attempt to price these derivatives, including Barone-Adesi-Whaley [11], Roll–Geske–Whaley [2], and the Black-Scholes-Merton (BSM) model [4], the latter of which I investigate during this research.

However sophisticated these models may be, they are claimed to be "mechanical" or "know-nothing" value systems. This is to say that prices they derive are based heavily on a few variables. In the case of the BSM model, implied volatility is a key factor, however it is not necessarily the best indicator of an option's value. While the BSM model is highly accurate for short-term contracts, a more sophisticated valuation may be required for longer-term positions. [6] [13]

As part of this project I seek to investigate the benefits and drawbacks of existing pricing models as well as to produce an extension of my own, evaluating it's performance.

Key Words: Multinomial Option Pricing, Binomial Option Pricing, Black-Scholes-Merton, Option Pricing, Implied Volatility Smile, Stochastic Parameters

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1 Introduction

A derivative is a financial instrument whose value is derived from the value of yet another asset, known as the underlying asset. Derivatives play an important role in modern finance, they allow for various financial activities such as risk management, speculation, and arbitrage [15].

Amongst the different types of derivatives, options hold a significant place in the markets. An option provides its holder with the right, but not the obligation, to buy (call) or sell (put) the underlying asset at a predetermined price, known as the strike price, before a specified date, known as the maturity or expiration date. This flexibility allows investors to perfect their investment strategies according to market conditions, personal risk adversity, and the risk preferences of the firm.

Risk Management (Hedging): The most prevalent form of risk management typically involves hedging. This is the strategy where an individual takes a second position in the market to offset potential losses in some initial investment. Option contracts allow investors to lock in prices without the obligation to buy or sell the underlying asset unless it benefits their position.

For example, protective puts - these involve buying put options for an asset that is already owned. The put option increases in value as the asset's price decreases, helping to offset losses in the asset's value.

Speculation: Speculation is where investors make bets on future market movements. Options are attractive for speculation because they offer leverage, meaning they allow speculators to gain a large exposure to a stock for a relatively small initial investment — the option premium (this is due to the fact that a single options contract covers more than 1 unit of the underlying asset). This can amplify both profits and losses. Speculators might buy call options if they anticipate the underlying asset will increase in price or buy put options if they expect the asset to decrease. The potential loss is limited to the premium paid (as one can simply choose not to execute the contract), but the potential gain can be substantial.

Arbitrage: Arbitrage using options involves taking advantage of pricing inefficiencies between related options markets or between an option and its underlying asset. Strategies like conversion and reversal arbitrage — where one buys the underlying asset and simultaneously invests in calls and puts to exploit small price discrepancies. Another arbitrage strategy is that of box spreads, which involve buying and selling a special combination of put and call contract spreads to create arbitrage opportunities, showcasing how options can be used to exploit market inefficiencies.

Market Makers: Market makers play an important role in options markets by providing liquidity, which means they ensure that there is always a buyer or seller available for options contracts. They continuously quote buy and sell prices for various options, market makers help to reduce price disparities and maintain market efficiency. They profit from the bid-ask spread — the difference between the buy and sell price. This function is vital in certain options markets where the volume of actively traded options are lower than when compared to stocks, and thus, without market makers, it becomes increasingly challenging for investors to execute orders quickly at a desired price.

These strategies highlight the flexibility of options providing valuable opportunities to manage and capitalize on financial risks.

1.1 Options contracts overview

Options contracts have several key features:

- **Underlying Asset**: This is the financial instrument on which the option's value is based. It can be any financial asset including stocks, commodities, currencies, indices, or even another derivative. Naturally, the movement in price of the underlying asset has a correlated effect on the option's price too.
- **Maturity or Expiration Date**: This is the date at which the option contract expires and becomes void. After this date, the option holder no longer has the right to exercise the option. They forgo any funds they've paid to own the option contract. Options that have a long time before they expire have a higher *time-value*, as it is more likely they will move *into the money* before expiration (relative to those due to expire very soon).
- **Strike Price**: Also known as the exercise price, this is the price at which the underlying asset can be bought or sold if the holder chooses to exercise the option. The strike price is predetermined at the time the option contract is created, and the contract's value is hence determined partly by the price of the underlying asset relative to the strike price. We look at this in further detail below when we cover pay-off diagrams.
- **Option Price or Premium**: This is the price that the option buyer pays to the option seller for the right to buy or sell the underlying asset. The option price is influenced by factors such as the aforementioned as well as, volatility of the underlying asset's price and interest rates.

Options contracts can be further classified based on their type and exercise style:

- **Call Options**: Call options give the holder the right to **buy** the underlying asset, at the strike price, before the expiration date. Call option holders profit when the price of the underlying asset rises above the strike price, as it means they can buy the underlying below the market price (and then sell to the market for a profit).
- **Put Options**: Put options give the holder the right to sell the underlying asset at the strike price within the specified period. Put option holders profit when the price of the underlying asset falls below the strike price, as it means they can sell the underlying above the market price (and then buy back from the market for a profit).
- **European Options**: European options can only be exercised **at** the expiration date. Holders of European options have no opportunity to exercise the option before expiration.
- **American Options**: American options can be exercised at **any time** before or on the expiration date. Holders of American options have the flexibility to exercise the option at any point during the option's lifespan.

Understanding the characteristics and mechanics of options contracts is essential for investors and traders to make informed decisions and manage risk effectively. [15]

N.B. Unlike the names may suggest, American and European options have no geographic restrictions with regards to the markets they are traded on and exist globally.

1.2 Motivation

This project represents the culmination of my three year long academic journey at Warwick, pursuing undergraduate studies, and aided via practical experience gained as a Quantitative Analyst intern at Goldman Sachs during the summer of 2023. I was motivated by a passion for finance, an interest in current affairs and a desire to expand my understanding of options pricing, this endeavor explores the intricacies of financial derivatives, in specific options.

Options have been prominent in financial markets due to their versatility and wide range of aforementioned uses. This research focuses on the nuances of options contracts, including underlying assets, maturity timelines, strike prices and more. Particularly, it focuses on discussing existing models - acting as a source of inspiration behind my own attempt at developing an options pricing model.

In summary, this research project represents a scholarly attempt at contributing to the understanding of options pricing through rigorous analysis and empirical investigation.

1.3 Objectives

I now set out the objectives I wish to achieve during this research project:

1. Understand historical and current background of option pricing models
2. Build knowledge to be able to intuitively understand the derivations of existing models
3. Research remedies for known pitfalls of existing models
4. Produce a model (or extension of an existing one)
5. Develop an automated testing framework for models
6. Evaluate existing models and my own one against a range of equities to evaluate performance

1.4 Option payoff diagrams

We focus on American options for this research, as it is the most commonly traded type of option. However, when modelling prices we often model European option prices and then convert them after the fact to the equivalent American prices.

Option payoff diagrams provide a visual representation of the potential profit or loss from holding an option position at a given point in time. These diagrams plot the option's payoff against the price of the underlying asset at some time t, illustrating how the option's value changes with different underlying asset prices.

Payoff diagrams vary depending on the type of option (call or put). They help investors analyze and visualize the risk-reward profile of option strategies and make strategic decisions based on their financial goals and risk tolerance. We call K the strike price of the option we hold, recall the Premium is how much we paid to own the option.

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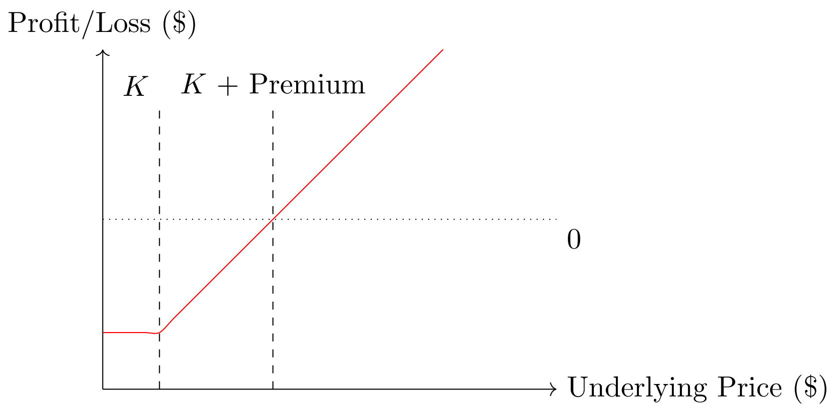


Figure 1: Call Option Payoff Diagram

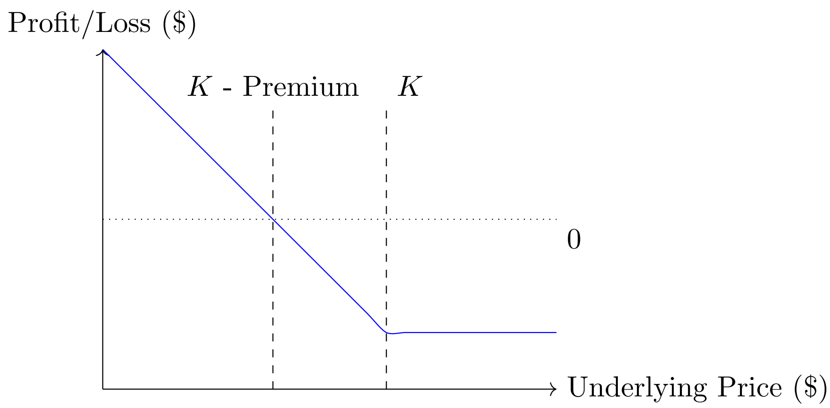


Figure 2: Put Option Payoff Diagram

For a call option:

- You make a profit if the price of the underlying asset at any point before expiration exceeds the strike price plus the premium paid for the option. This is because you have the ability to exercise the option early and capture the difference between the stock price and the strike price.
- If you do not exercise the option before expiration, you forgo the premium you paid to own the option in the first place.

For a put option:

- You make a profit if the price of the underlying asset at any point before expiration falls below the strike price minus the premium paid for the option. Similar to call options, you can exercise the put option early and capture the difference between the strike price and the stock price.
- As with call options, if you do not exercise the option before expiration, you forgo the premium you paid to own the option in the first place.

These characteristics of profit or loss for options are reflected in the corresponding payoff diagrams.

Explanation of some more options terminology:

- *In the money* (ITM): An option is *in the money* if exercising it results in a profit. For a call option, this means the underlying asset's price is above the strike price. For a put option, it means the underlying asset's price is below the strike price.
- *Out of the money* (OTM): An option is *out of the money* if exercising it would result in a loss. For a call option, this means the underlying asset's price is below the strike price. For a put option, it means the underlying asset's price is above the strike price.
- *At the money* (ATM): An option is *at the money* when the underlying asset's price is equal to the strike price, meaning that exercising the option would neither gain nor lose money.

We cover some key properties and parameters of options below.

1.5 Factors affecting option prices

We briefly touched on some factors that influence the price of options, we look at a few more and in further detail here:

- **Underlying Asset Price**: The price of the underlying asset has a direct impact on the value of options. For call options, higher underlying asset prices increase the option's value, while for put options, lower underlying asset prices increase the option's value.
- **Volatility**: Volatility measures the degree of fluctuation in the price of the underlying asset. Higher volatility leads to higher option prices due to increased likelihood of the option contract moving *into the money.*
- **Time to Expiration**: The longer the time to expiration, the higher the option price, as there is more time for the underlying asset to move *into the money.*
- **Interest Rates**: Interest rates affect the value of future profits and losses associated with options. Higher interest rates lead to lower option prices, while lower interest rates lead to higher option prices. We look at this in more detail when we discuss the risk-free rate and risk-free assets.
- **Dividends**: For stocks paying dividends, the dividend date and dividend amount can impact option prices, especially for call options. This is due to the fact that dividends have an effect on the underlying asset's price, and hence a knock-on effect on the options.

These factors interact overlapping one another, as a result of their dynamic nature. We use them all in conjunction to determine option prices, making options pricing a complicated process.

1.6 Fundamental theorem of asset pricing review

To understand the intuition behind pricing any financial derivative, we employ the Fundamental Theorem of Asset Pricing (FTAP). The theorem provides conditions that keep financial markets 'arbitrage-free'. [21]

- **The First Fundamental Theorem of Asset Pricing:** A discrete market on a discrete probability space $(\Omega, \mathcal{F}, P)$ is arbitrage-free if, and only if, there exists at least one risk-neutral probability measure that is equivalent to the original probability measure, P.
- **The Second Fundamental Theorem of Asset Pricing:** An arbitrage-free market (S, B) consisting of a collection of stocks S and a risk-free bond B is complete if and only if there exists a unique risk-neutral measure that is equivalent to P and has the numeraire B.

[24]

More intuitively, to avoid arbitrage, and hence to capture the true price of a derivative we generally seek to value a derivative at its predicted value at maturity, less the return on a perfectly risk-free asset. We now look at risk-free assets, and understand why interest rates effect the value of option contracts.

1.7 Risk free assets and the risk free rate review

Ensuring we account for the fundamental theorem of asset pricing means that when valuing a financial asset, we always need to consider risk - the measure of risk is almost always relative to another asset. When putting money into an option, we naturally consider the opportunity cost of putting the same money into a completely risk-free asset. In reality, there are no assets that are truly risk free but we can get suitably close.

The risk free rate, is the rate of yield you would get on a theoretically risk-free asset.

For options listed on the American market, we usually use the yield rate of the 10-year US Government Treasury. This represents an asset that has a guaranteed return, posing no risk to the investor.

However, the same does not hold for all other markets - if a government is deemed "untrustworthy" (by the market's investors), and poses default risk to the investor, we cannot use the sovereign bond rate as it is no longer considered risk-free. We will look into how to combat this in future, but initial development will be focused on options listed on the American markets. [9]

1.8 Modern research and industry secrecy

Modern research in options pricing models takes place at a level of complexity usually unreachable by those who aren't working at a postdoctoral standard. However, a notable aspect that causes difficulty in this field is the information gap between academic research and industry advancements. While academic institutions contribute significantly to theoretical frameworks and foundational concepts, major breakthroughs in options pricing models often emerge from research conducted within industry settings.

In contrast to academia, where research findings are typically distributed via publications and conferences, the landscape of industrial research presents unique challenges. Due to the proprietary nature of financial innovations and the competitive advantage they provide, the bulk of advancements in options pricing models remain shielded from the public domain. Industry researchers often guard their findings by safeguarding them behind layers of non-disclosure agreements (NDAs) and confidentiality clauses. This is primarily due to the competitive advantage it gives them over other firms, particularly in the financial services industry - where an advantage directly correlates to increased profits.

It is these dynamics that I have had to navigate and acknowledge during my research. I have remained cognizant of the fact that the research at an industry standard is extremely powerful and delivers the most accurate insights available, but I embarked on this project out of an interest to enhance the information **publicly** available.

It is for these same reasons a portion of my learning and research has come form nontraditional sources, in the form of guest lecture recordings and articles as well as real life conversation with those involved with financial research.

1.9 Literature review

Before beginning any attempts at pricing options contracts, it was essential to understand the current academic landscape for this area of research. I looked at various models, all very unique and each with their own strengths, allowing me to better gauge which avenues were open for improvement upon existing implementations. This part of reading was at a high-level, to develop familiarity with current publications - an in depth literature review was conducted for the specific models I chose to investigate, and is covered further on.

1.9.1 Black-Scholes Model

[5] The Black-Scholes Model, introduced by Fischer Black and Myron Scholes in 1973, revolutionized the field of financial economics by providing a closed-form solution for the pricing of European-style options.

This model assumes that the volatility of the underlying asset is constant and that the asset price follows a geometric Brownian motion, offering a straightforward framework that balances the risk of holding the underlying asset against the potential payoff from exercising the option.

The strengths of the Black-Scholes model include its (relative) simplicity, which aids in the clarity of the mathematical framework, requiring only five input parameters (current stock price, strike price, time to expiration, risk-free rate, and volatility).

However, the model is limited by several critical assumptions that do not hold in real-world markets: it assumes constant volatility and a normal distribution of returns, overlooking the empirical reality of fluctuating volatility and the occurrence of extreme market movements (known as fat tails).

Additionally, it is only applicable to European options that can be exercised only at expiration, does not account for dividends, and assumes a constant risk-free rate. While the Black-Scholes model provides a robust framework under certain conditions, its practical application is often limited, prompting financial professionals to adjust the original framework or employ alternative models to better capture market movements and the complexities of financial instruments. One such model is the Black-Scholes-Merton Option Pricing model.

1.9.2 Binomial Option Pricing Model

[8] The Binomial Option Pricing Model, proposed by Cox, Ross, and Rubinstein in 1979, offers a discrete-time approach to pricing options through the construction of a binomial tree that represents

possible price movements of the underlying asset over time.

Unlike the Black-Scholes Model, this model can handle American-style options and is more flexible in modeling complex market dynamics, making it particularly valuable for American options where exercise may occur at multiple points rather than only at expiration.

However, a significant limitation of the Binomial Model is its requirement to specify the number of time steps, which can make the model computationally intensive, especially as the number of steps increases to capture more accurate price movements and option values.

1.9.3 Trinomial Option Pricing Model

[26] Similar to the binomial model, this approach introduces a trinomial framework where there are three possible price movements at each time step instead of two. This enhancement offers a more accurate representation of price dynamics, providing a finer granularity in modeling the potential paths that the underlying asset's price might take over time.

However, this added accuracy comes at the cost of increased computational complexity, as the expansion in possible outcomes at each step significantly enlarges the tree structure, thus requiring more computational resources to evaluate options accurately.

Whilst this model provides an improvement in accuracy over binomial model, the huge increase in computational resources makes it essentially negligible.

1.9.4 Heston Model

[14] The Heston Model, introduced by Steven Heston in 1993, extends the Black-Scholes framework by incorporating stochastic volatility, thereby allowing volatility to vary over time. This model is built using fundamental techniques such as capturing features like volatility clustering and mean reversion, which are prevalent in financial markets.

The model's ability to capture time-varying volatility makes it particularly useful for pricing options on assets whose volatility is not constant. However, the inclusion of stochastic volatility introduces additional parameters into the model, which not only increases its computational complexity but also presents calibration challenges as these parameters must be accurately estimated to reflect market dynamics effectively. It also adds another layer of failure when diagnosing discrepancies in our model.

1.9.5 SABR Model

[12] The Stochastic Alpha Beta Rho (SABR) model, introduced by Hagan et al. in 2002, serves as a robust framework for pricing options, particularly for assets characterized by high volatility. This model excels in modeling both the volatility smile and skew observed in option markets, making it highly effective at capturing the dynamics of volatility surfaces.

It is especially prevalent in the interest rate derivatives markets, where these characteristics are often pronounced. However, the SABR model's sophistication requires the use of complex numerical methods for its calibration and simulation, posing challenges in practical implementations due to the computational demands and the expertise required to execute these techniques accurately.

1.9.6 Machine Learning-Based Models

Option pricing models developed using machine learning techniques harness historical data to learn complex patterns and relationships, offering a significant advancement over traditional models.

These machine learning-based models are particularly specialised at capturing non-linearities and interactions between variables that traditional methods may not detect, making them particularly useful for pricing options on assets characterized by complex price dynamics. However, the reliance on large datasets for training and the opaque nature of machine learning algorithms can make these models challenging to interpret and validate, presenting hurdles in terms of transparency and reliability in their practical application.

1.9.7 Deep Learning-Based Models

Options pricing models developed using deep learning techniques, such as neural networks, are specifically built to handle large volumes of data and designed to automatically extract high-level features. These models excel at capturing intricate patterns in option pricing data and are capable of adapting to changing market conditions, which is crucial for accurate financial modeling.

However, deep learning models often face criticism for their "black box" nature, which obscures the underlying decision-making processes and makes it difficult to understand the reasoning behind their predictions.

This lack of transparency can complicate the assessment of the models' robustness and reliability, presenting challenges for their acceptance and use in critical financial decision-making environments.

1.10 Choice of models

Having understood at a surface level, the advantages and disadvantages of the aforementioned models, I decided to select a few models for in-depth research, namely those that had varied fundamental methods of implementation, and each with their own unique strengths and weaknesses. I also carefully chose models that allowed me to both reach beyond my formal academic learning, whilst making sure not to overwhelm myself in the time constraints of the project.

Naturally, I picked the Black-Scholes model for further investigation, specifically the Black-Scholes-Merton model (an extension that accounts for dividend yield). The reason for this choice, is due to the fact the BSM model underpins almost every successful implementation of options pricing. Understanding the BSM model would put me in a position to think more intuitively about the problem at hand.

Secondly, I also chose to use the Binomial Options Pricing model, it has been cited many times as offering incredible flexibility - a property that is particularly important for when I choose to implement various ideas I'd had. This flexibility was particularly important, as it allowed for me to take characteristics from other existing models and implement them in a way that maintained the benefits of both the Binomial model as well as the one being integrated into it.

Both models also required a level of knowledge, that I could independently gain, to work effectively with them. The same unfortunately did not hold for ML and DL based models. Nevertheless, I was curious as to how I could integrate Neural Networks into an options pricing model and briefly explored it as an extension to the research I've covered.

2 Testing framework setup

2.1 Overview

The testing framework is the pipeline that I constructed to make it faster and easier for me to analyse and compare multiple models with the least interaction, in the form of writing new code, required. The framework ensured I could define a single function for each model, and essentially "plug" it in to my framework to test it with consistently the same inputs and analysis methods.

2.2 Choosing appropriate language

To easily implement ideas and concepts, as well as options pricing models, during my research, I decided to use Python as my primary language. This required me to weigh up the costs and benefits of the language:

Pros:

- Ease of Implementation: Python's clean and simple syntax makes it easy to translate ideas and concepts into code, facilitating quick prototyping and experimentation in financial modeling.
- Rich Ecosystem: Python boasts a vast ecosystem of libraries and frameworks tailored for finance, including NumPy, Pandas, and SciPy, providing powerful tools for data manipulation, analysis, and modeling.
- Community Support: Python enjoys a large and active community of developers, particularly in the field of finance. This community provides extensive documentation, tutorials, and forums for assistance.
- Visualization Tools: Python offers robust visualization libraries like Matplotlib and Seaborn, enabling the creation of insightful charts and graphs to communicate financial data and model outputs effectively.

Cons:

- Performance: While Python is highly versatile, it may not always offer the same level of performance as lower-level languages like C++ or Fortran, particularly for computationally intensive tasks in financial modeling.
- Global Interpreter Lock (GIL): Python's GIL can limit concurrency and parallelism in multi-threaded applications, potentially impacting performance in scenarios requiring heavy parallel processing, although this limitation can be mitigated using alternative concurrency models or external libraries.
- Dependency Management: Managing dependencies and package versions can sometimes be challenging in Python projects, especially when working with numerous libraries and dependencies with complex inter-dependencies.

Nevertheless, being experienced in using python for various applications, I decided to acknowledge the potential pitfalls for when I implement the framework. For reasons regarding visualisation, during development I used a Jupyter Notebook and for implementation I wrote Python files distributed to a cluster that we will discuss later.

2.3 Data sourcing

Many platforms were considered when choosing a specific source for options data, however they fell into two main categories. First, free and imperfect data and secondly, paid but highly accurate data. The best example of both and their costs and benefits are listed below.

2.3.1 Yahoo Finance:

Pros:

- Free Data: Yahoo Finance provides historical options data for free.
- User-Friendly Interface: It offers a user-friendly interface for browsing and downloading data.
- Variety of Instruments: Yahoo Finance covers a wide range of options contracts across various assets.

Cons:

- Limited Depth: Depth of historical options data may be limited.
- Data Quality: While generally reliable, occasional discrepancies could occur.
- Limited Customization: Yahoo Finance may offer limited customization options for data retrieval.

2.3.2 CBOE (Chicago Board Options Exchange):

Pros:

- Official Exchange Data: CBOE provides official exchange data with high accuracy.
- Comprehensive Coverage: It covers a wide range of options contracts traded on its exchange.
- Depth of Data: CBOE offers detailed historical options data.

Cons:

- Cost: Access to CBOE's historical options comes with a cost.
- Complexity: CBOE's data retrieval interfaces may be complex.
- Limited Accessibility: Official exchange data may not be easily accessible.

For many of the paid services, the price was in excess of $10,000 per month as a result of their target consumers being institutions. Also, the scope of the information they were offering to provide was in excess of the investigation carried out during this project. For that reason, free sources were considered as the primary source of data.

2.4 Underlying Data Acquisition

Whilst all of the free sources had similar accuracy and data clarity Yahoo Finance was chosen due to the availability of a very documented and highly flexible python module that let me interface directly with data, *yfinance*.

The development of the options pricing model involved careful data processing to ensure accuracy and reliability. We now delve into the technical details of the code and discuss the motivations behind each step:

The initial step in building the options pricing model was to acquire historical price data for the underlying asset. This was accomplished by utilizing the Yahoo Finance API, a popular and reliable source for financial data. The choice of minute intervals for data collection was driven by the need for high-frequency data, enabling finer granularity in modeling price movements.

```
# Create a yfinance Ticker object for the specified underlying asset
underlying = yf.Ticker(underlying_ticker)

# Obtain historical price data at minute intervals
price_data = underlying.history(start=start_date, end=end_date, interval="1m")[["
    Open", "Close"]]
```

2.5 Data Preprocessing

Once the historical price data was obtained, preprocessing steps were undertaken to refine the data for further analysis. The calculation of the average price, derived from the "Open" and "Close" prices, served to provide a representative value for each trading period. ("Open" and "Close" prices represent the price of the asset at the start and end of the given time interval, depending on data resolution). Standardizing the time zone information to a consistent format was essential to ensure uniformity in data representation, facilitating accurate analysis and comparison across different time periods.

```
# Calculate the average price and set the time zone to None
price_data["Price"] = (price_data["Open"] + price_data["Close"]) / 2
price_data.index = price_data.index.tz_localize(None)
```

2.6 Volatility Calculation

Volatility is a crucial parameter in options pricing models, as it reflects the magnitude of price fluctuations over time. To compute volatility, a rolling window approach was employed, utilizing the historical price data collected. By measuring the standard deviation of price returns within a specified window, the resulting volatility information offered insights into the asset's price variability. Normalizing volatility data over a five-year period allowed for a comprehensive assessment of historical price movements, enabling more informed pricing decisions.

```
# Obtain volatility data for the past 5 years at daily intervals
volatility_data = pd.DataFrame([])
current_date = datetime.today() - timedelta(days=30)

while current_date <= end_date:
    end_day = current_date + timedelta(days=1)
    with contextlib.suppress(Exception):
        day_data = underlying.history(start=current_date, end=end_day, interval="1
    m")
```

```
    volatility_data = pd.concat([volatility_data, day_data])

    current_date += timedelta(days=1)

# Calculate and add volatility information to the price_data
scaling_factor = volatility_data['Close'].pct_change().abs().mean()
volatility_data['Close'] = pd.to_numeric(volatility_data['Close'], errors='coerce')
price_data["Volatility"] = (volatility_data['Close'].pct_change().pow(2).rolling(window=1).sum().pow(0.5) / scaling_factor).mean()
```

Explanation: The volatility calculation involves several steps to ensure an accurate representation of price variability over time.

1. **Data Collection**: The volatility data is obtained from historical price data collected over the past 5 years at daily intervals.

2. **Rolling Window Approach**: A rolling window approach is utilized to compute volatility. This involves calculating the standard deviation of price returns within a specified window, which captures short-term fluctuations in price movements.

3. **Scaling Factor Calculation**: The scaling factor is computed as the mean absolute percentage change in closing prices. This factor is used to normalize the volatility data and ensure consistency in measurement across different assets and time periods.

4. **Normalization**: The volatility data is normalized using the scaling factor to adjust for variations in price levels and facilitate meaningful comparison.

This is a rudimentary approach at accounting for volatility, and is sufficient for a good starting point in option price modelling. However, more sophisticated methods do exist, and is an area of research that is covered later on and implemented in my final model.

2.7 Dividend Data Integration

Incorporating dividend data into the options pricing model is essential for accurately assessing the asset's total return potential. Historical dividend data spanning the past ten years was integrated into the model to capture long-term dividend trends. The calculation of dividend yield, expressed as a percentage of the average price, provided valuable insights into the asset's income-generating potential. By factoring in dividend information, the model could more accurately estimate the asset's total return, enhancing its predictive capabilities, as an asset's return on investment directly correlates to the value of holding said asset.

```
# Obtain dividend data for the past 10 years at daily intervals
dividend_start_date = datetime.strptime(start_date, "%Y-%m-%d") - timedelta(days=365 * 10)
dividend_data = underlying.history(start=dividend_start_date, end=end_date, interval="1d")

# Calculate and add dividend information to the price_data
price_data["Dividend"] = (underlying.dividends.iloc[-1] / np.mean(dividend_data["Close"])) * 100
```

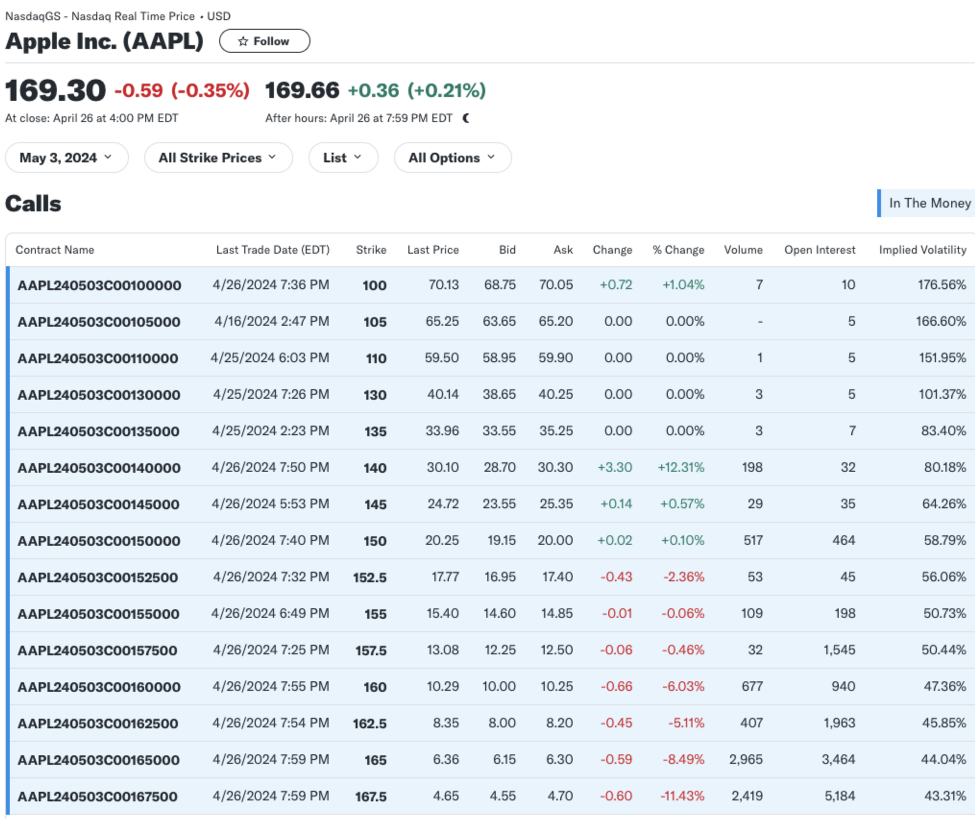
Explanation: The calculation of dividend yield involves several steps to accurately estimate the income-generating potential of the underlying asset.

1. **Data Collection**: Historical dividend data spanning the past 10 years is obtained at daily intervals.

2. **Dividend Yield Calculation**: The dividend yield is computed as the ratio of the latest dividend payout to the average price over the specified time period. This provides insights into the asset's income generation relative to its price level, allowing for a more comprehensive assessment of total return potential.

We calculate volatility and dividend yield in a simplistic fashion here. However, the final framework has a much more sophisticated calculation that we will explore in further detail.

2.8 Options Data Acquisition


NasdaqGS - Nasdaq Real Time Price • USD

Apple Inc. (AAPL) ☆ Follow

169.30 -0.59 (-0.35%) **169.66** +0.36 (+0.21%)

At close: April 26 at 4:00 PM EDT After hours: April 26 at 7:59 PM EDT

May 3, 2024 | All Strike Prices | List | All Options

Calls In The Money

Contract Name	Last Trade Date (EDT)	Strike	Last Price	Bid	Ask	Change	% Change	Volume	Open Interest	Implied Volatility
AAPL240503C00100000	4/26/2024 7:36 PM	100	70.13	68.75	70.05	+0.72	+1.04%	7	10	176.56%
AAPL240503C00105000	4/16/2024 2:47 PM	105	65.25	63.65	65.20	0.00	0.00%	-	5	166.60%
AAPL240503C00110000	4/25/2024 6:03 PM	110	59.50	58.95	59.90	0.00	0.00%	1	5	151.95%
AAPL240503C00130000	4/25/2024 7:26 PM	130	40.14	38.65	40.25	0.00	0.00%	3	5	101.37%
AAPL240503C00135000	4/25/2024 2:23 PM	135	33.96	33.55	35.25	0.00	0.00%	3	7	83.40%
AAPL240503C00140000	4/26/2024 7:50 PM	140	30.10	28.70	30.30	+3.30	+12.31%	198	32	80.18%
AAPL240503C00145000	4/26/2024 5:53 PM	145	24.72	23.55	25.35	+0.14	+0.57%	29	35	64.26%
AAPL240503C00150000	4/26/2024 7:40 PM	150	20.25	19.15	20.00	+0.02	+0.10%	517	464	58.79%
AAPL240503C00152500	4/26/2024 7:32 PM	152.5	17.77	16.95	17.40	-0.43	-2.36%	53	45	56.06%
AAPL240503C00155000	4/26/2024 6:49 PM	155	15.40	14.60	14.85	-0.01	-0.06%	109	198	50.73%
AAPL240503C00157500	4/26/2024 7:25 PM	157.5	13.08	12.25	12.50	-0.06	-0.46%	32	1,545	50.44%
AAPL240503C00160000	4/26/2024 7:55 PM	160	10.29	10.00	10.25	-0.66	-6.03%	677	940	47.36%
AAPL240503C00162500	4/26/2024 7:54 PM	162.5	8.35	8.00	8.20	-0.45	-5.11%	407	1,963	45.85%
AAPL240503C00165000	4/26/2024 7:59 PM	165	6.36	6.15	6.30	-0.59	-8.49%	2,965	3,464	44.04%
AAPL240503C00167500	4/26/2024 7:59 PM	167.5	4.65	4.55	4.70	-0.60	-11.43%	2,419	5,184	43.31%

Figure 3: Apple (AAPL) Call Option Chain on Yahoo Finance [30]

Once the object is initialized, we proceed to extract options data based on the initial parameters. Notably, we constrain our data to a single day due to limitations defined by the agreement for free use of Yahoo Finance.

```python
def get_option_data(underlying_ticker, option_exp, option_type, option_strike, start_date, end_date):
    # Construct option symbol
    option_symbol = f"{underlying_ticker}{option_exp[2:4]}{option_exp[5:7]}{option_exp[8:10]}{option_type}00{option_strike}000"

    # Print option symbol
    print(option_symbol)
```

```python

    # Create yfinance Ticker object for the option
    option_ticker = yf.Ticker(option_symbol)

    # Retrieve option data
    option_data = option_ticker.history(start=start_date, end=end_date, interval="1m")[["Open", "Close"]]
    option_data["Price"] = (option_data["Open"] + option_data["Close"]) / 2
    option_data["Availability"] = len(option_data.index) / 389
    option_data.index = option_data.index.tz_localize(None)
    option_data["Date"] = option_data.index

    # Display and save option data
    print(option_ticker)
    display(option_data)

    return option_ticker, option_data

# Get call option data
option, option_data = get_option_data(underlying_ticker, option_exp, option_type, option_strike, start_date, end_date)
```

Explanation: The function `get_option_data` is responsible for extracting options data based on the specified parameters.

1. **Option Symbol Construction**: The option symbol is constructed using the underlying ticker symbol, option expiration date, option type (call or put), and option strike price. This symbol uniquely identifies the option contract for when a request is sent to the Yahoo Finance API endpoint.

2. **Option Ticker Creation**: A yfinance Ticker object is created for the option using the constructed option symbol.

3. **Option Data Retrieval**: Historical price data for the option is retrieved from the Yahoo Finance API, constrained to the specified start and end dates at a resolution of 1 minute intervals.

4. **Data Processing**: The option data is processed to calculate the average price and availability. The average price is computed as the average of the "Open" and "Close" prices. The availability metric represents the proportion of available data points relative to the total possible data points in a trading day (389 minutes for U.S. stock market).

5. **Display and Return**: The option data, along with the option ticker object, is displayed and returned for further analysis.

2.9 Data Imperfections Handling

Due to imperfect data from Yahoo Finance, we also initialize a column named "Availability". This is because, even on a minute-by-minute resolution (relatively coarse), we still have missing data. Hence, we preserve the availability of the data to gauge the effect on the accuracy of the various models we will test. Namely, the performance of the models on options with high trading volumes compared to the entire dataset.

See Figure 4 overleaf.

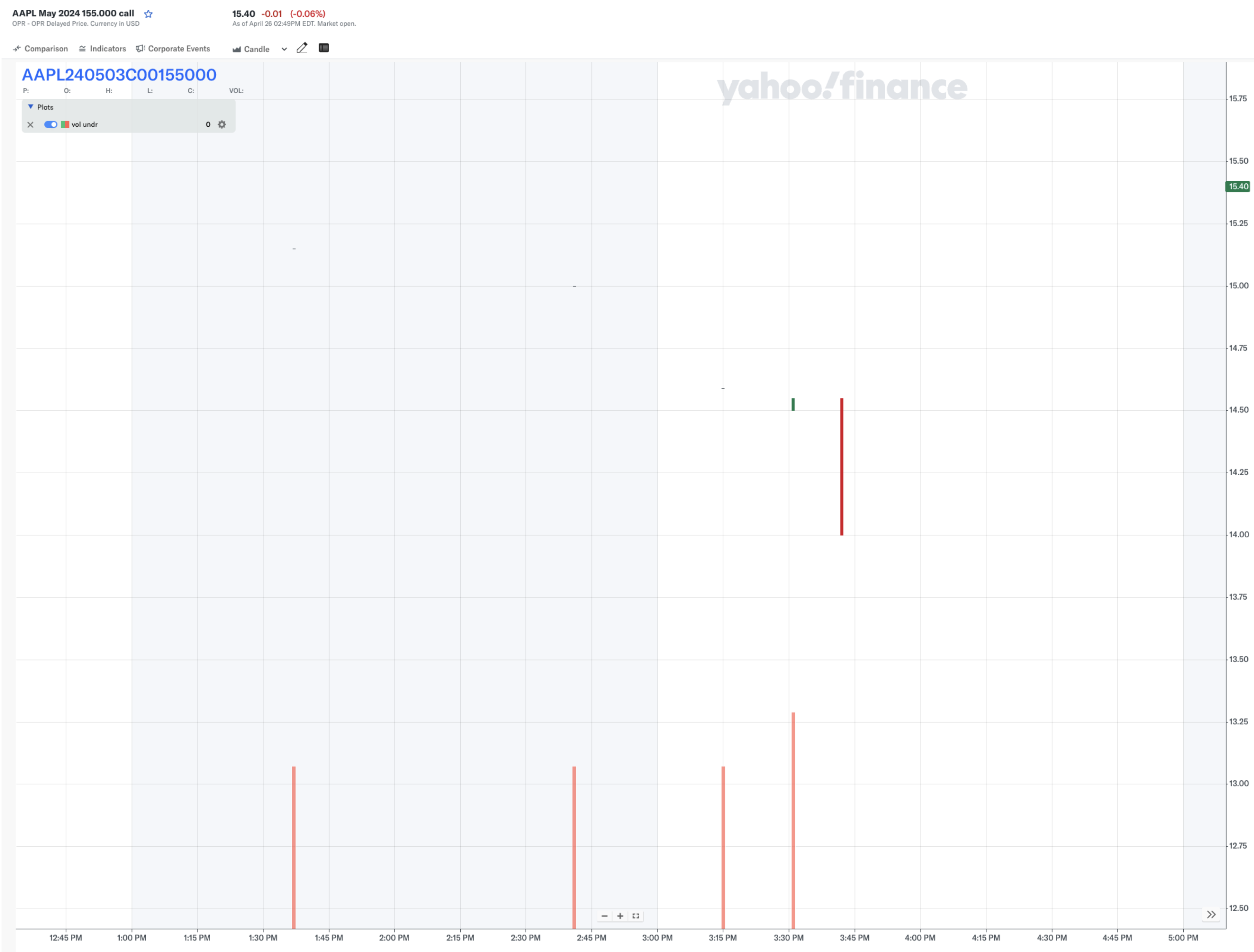


Figure 4: AAPL Call Option showing low trading volume [30]

2.10 Further Data Handling

We use the same technique to also pull data for the Risk Free Rate (can be intuitively thought of as the Central Bank Rate or Government Interest Rate), this can be seen in the full code file in the appendix, and will not be repeated here given it's similarity to the aforementioned data retrieval methods.

2.11 Example outputs

Below is an example of the output of the code discussed thus far.

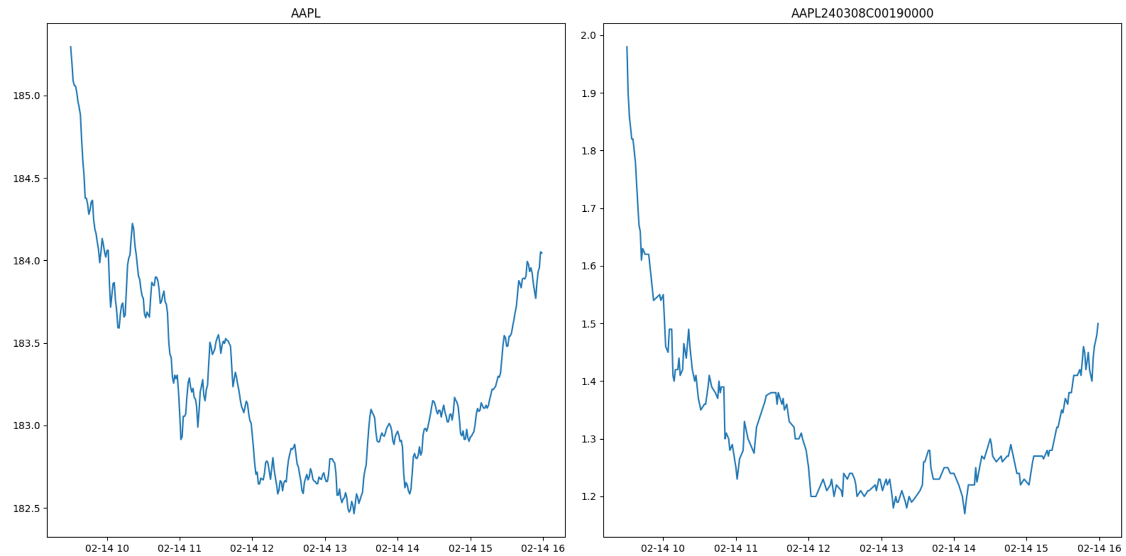


Figure 5: AAPL stock vs. Option data

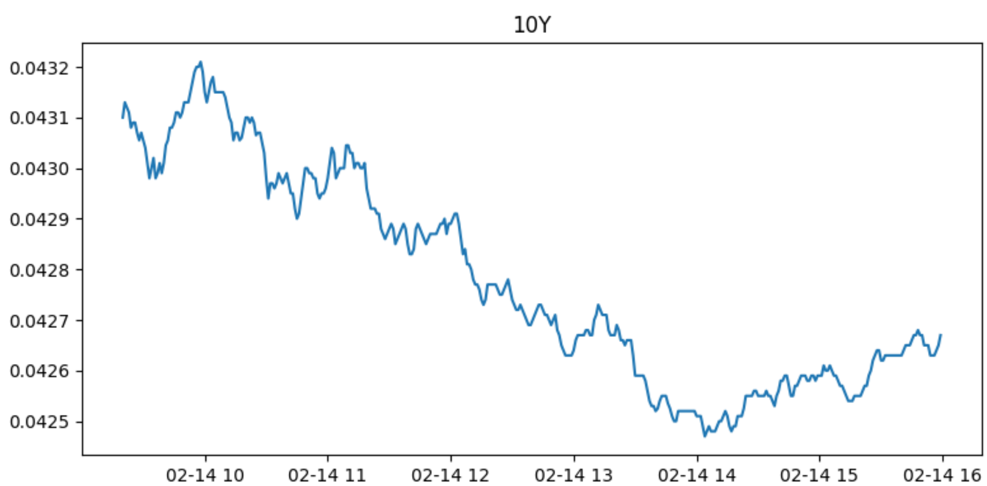


Figure 6: US 10-Year Treasury Yield Rate data

2.12 Model accuracy analysis

We do not include the model implementation steps of the framework as we will look at each one in detail later on. However, we take a look at how we analyse the performance of each model.

2.12.1 Choice of accuracy metrics

- **Mean Absolute Difference (MAD) [1]:**
  - Provides an absolute measure of the average magnitude of errors between predicted and actual option prices.
  - Easy to interpret and understand, representing the average discrepancy between predicted and actual values in the same units as the data.
- **Mean Percentage Change Difference (MPCD) [1]:**
  - Calculates the relative measure of the average magnitude of errors, expressed as a percentage of the actual option prices. This quantifies the error in terms of the proportion to actual values, offering a normalized measure that is particularly useful when comparing performance across data sets of varying scales or magnitudes.
  - Provides a straightforward interpretative metric that contextualizes the size of errors relative to the actual prices, thus allowing for a comparative assessment across different markets or asset classes where absolute prices differ significantly.
- **Root Mean Squared Error (RMSE) [29]:**
  - Similar to MAE but penalizes larger errors more heavily due to squaring.
  - Useful for evaluating the overall model accuracy by capturing both systematic and non-systematic errors.
- **Correlation Coefficient [17]:**
  - Measures the strength and direction of the linear relationship between predicted and actual option prices.
  - Provides insights into the model's ability to capture the underlying patterns in the data.

The use of these metrics as a standard of comparison across all of my models yielded insights that clearly highlighted the strengths and weaknesses of the models quantitatively.

2.12.2 Implementation

Having run the model on our data, we compare the model's outputs to the true value of options prices on that day. Below you can see the code that does this, particularly with the results of the Black-Scholes-Merton (BSM) model.

```python
# Merge the option data with the BSM values
eval_data = option_data[["Date", "Price"]].merge(underlying_data[["Date", "BSM"]],
    on=["Date"])

# Calculate evaluation metrics for BSM model
mae = np.mean(np.abs(eval_data["Price"] - eval_data["BSM"]))
rmse = np.sqrt(mean_squared_error(eval_data["Price"], eval_data["BSM"]))
r2_abs_diff = r2_score(eval_data["Price"], eval_data["BSM"])
```

```
correlation_coefficient, _ = pearsonr(eval_data["Price"], eval_data["BSM"])

# Calculate percentage changes, handle infinite values, and fill NaNs
eval_data["Price (%chng)"] = eval_data["Price"].pct_change().replace([np.inf, -np.
    inf], np.nan).fillna(0)
eval_data["BSM (%chng)"] = eval_data["BSM"].pct_change().replace([np.inf, -np.inf
    ], np.nan).fillna(0)
r2_pc_diff = r2_score(eval_data["Price (%chng)"], eval_data["BSM (%chng)"])

print(f"MAE: {mae}\n"
      f"RMSE: {rmse}\n"
      f"R^2 Abs Diff: {r2_abs_diff}\n"
      f"R^2 %chng Diff: {r2_pc_diff}\n"
      f"Correlation Coefficient: {correlation_coefficient}")
```

Explanation: The code snippet evaluates the model's performance by comparing its output (in this case, Black-Scholes-Merton values) to the true option prices for a given day.

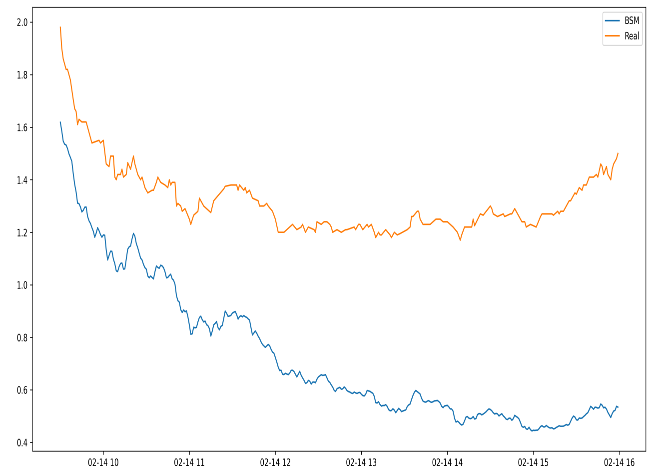
1. **Data Merging**: The option data, including the date and price, is merged with the Black-Scholes-Merton values calculated earlier for the corresponding dates.

2. **Evaluation Metrics Calculation**: Several evaluation metrics are calculated to assess the model's performance, as mentioned above.

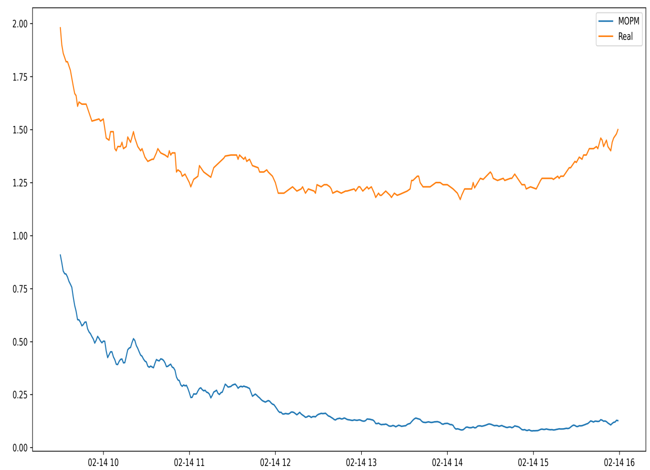
3. **Handling Data Transformations**: Percentage changes are calculated for both the true prices and Black-Scholes-Merton values to assess their relative performance. Infinite values and NaNs are handled appropriately to ensure accurate metric computation.

4. **Output**: The calculated evaluation metrics are printed to the console for analysis and interpretation, as well as a customised PDF report for each analysis for distribution and ease of access. An example of this PDF can be seen in the following pages.

Model Analysis



Black-Scholes-Merton (BSM) vs. Real - AAPL240308C00190000



Multinomial Option Pricing Model (MOPM) vs. Real - AAPL240308C00190000

Statistical Results

Evaluation Results:
BSM_Mean_Abs_Diff: 0.5784932096777813
BSM_RMSE: 0.6113289713290142
BSM_Correlation_Coefficient: 0.7952652150100457
BSM_Mean_Percent_Change_Diff: 0.012575473751197963
------------: ------------
MOPM_Mean_Abs_Diff: 1.0966372569916414
MOPM_RMSE: 1.100646218334432
MOPM_Correlation_Coefficient: 0.8608852394267758
MOPM_Mean_Percent_Change_Diff: 0.03072012820193903
-----------: -----------
BOPM_Mean_Abs_Diff: 0.8964912188571877
BOPM_RMSE: 0.9026651387643865
BOPM_Correlation_Coefficient: 0.8101092918931387
BOPM_Mean_Percent_Change_Diff: 0.015456868736817132

2.13 Clustered computing and performance optimisation

To provide a comprehensive review of each model, the models were analysed on over 150 tickers, and on average 20 days of data per ticker. In total, the total number of iterations per model was just shy of 320,000 samples. The samples were spread amongst various market sectors, albeit limited to those with high option activity, they were also differing in underlying volatlity to test the models in various scenarios.

The large sample set meant the process of analysing the models was an extremely resource intensive process, primarily CPU and Network bound (relating to the downloading of data from Yahoo Fiance and the iterative nature of many Options Pricing Models) The two key changes that were made when automating the analysis process were to tackle the bottlenecks produced by the aforementioned resources - firstly, Numpy was replaced by Cupy - a module that produces the same functionality as Numpy, but makes use of GPU-Acceleration for arithmetic operations. Secondly, the 150 ticker were split into 4 sets and exported to a cloud cluster of 4 servers to parallelise the process of analysis. CPU operations were also threaded asynchronously to spread the load across multiple cores, again furthering aiding my attempt of parallelisation.

2.13.1 Cluster Overview

For hosting the cluster, initially I used my personal homelab with Proxmox Hypervisor. A hypervisor, also known as a virtual machine monitor (VMM), is a software or firmware layer that enables the creation and management of virtual machines (VMs) on a physical host machine. Its primary function is to abstract and virtualise the underlying hardware resources of a physical computer, allowing multiple operating systems (OSes) or instances to run concurrently on the same physical hardware. This host machine was configured with 128GB Memory and 32 core threaded Intel Xeon Prcoessor. Upon testing, the network bottleneck was persistant due to my home only having a 100Mbps internet connection that could not be fully saturated by the processes. For this reason the final implementation was conducted on a remote cloud platform called Linode. I discuss Linode in more detail, below.

2.13.2 Linode Instance Overview

Linode is a cloud service platform, much like Azure from Microsoft or AWS from Amazon, however it focuses primarily on providing Virtual Cloud Machines at a low cost - making it perfect for a student's budget. Below you can see a breakdown, from the Linode website, of the instance I used for each machine in my cluster.

Use of the cluster was akin to using the Linux terminal on any personal or DCS machine and so I was faced with no difficulties in making full use of the resources made available to me, a product of my familiarity with the latter systems.

The overall time improvement from using clustering and parallelisation yielded a reduction in running time from 36 hours to 1.5 hours for all 80,000 samples. Factors such as efficiency and speed were not the primary scope of optimisation for this project, and any improvements that I made were made for ease of use and debugging during development as opposed to a focus on research.

In industry, the models discussed in this report are implemented on proprietary infrastructure using proprietary scripting languages. They are very purpose made, and have their main priority being efficiency of resource use and hence speed. Given my research is an extension of existing models,

Plan	Monthly	Hourly	RAM	CPUs	Storage	Transfer	Network In / Out
Premium 4 GB	$43	$0.0645	4 GB	2	80 GB	4 TB	40 Gbps / 4 Gbps
Premium 8 GB	$86	$0.129	8 GB	4	160 GB	5 TB	40 Gbps / 5 Gbps
Premium 16 GB	$173	$0.2595	16 GB	8	320 GB	6 TB	40 Gbps / 6 Gbps
Premium 32 GB	$346	$0.519	32 GB	16	640 GB	7 TB	40 Gbps / 7 Gbps
Premium 64 GB	$691	$1.0365	64 GB	32	1280 GB	8 TB	40 Gbps / 8 Gbps
Premium 96 GB	$1037	$1.5555	96 GB	48	1920 GB	9 TB	40 Gbps / 9 Gbps
Premium 128 GB	$1382	$2.073	128 GB	50	2500 GB	10 TB	40 Gbps / 10 Gbps
Premium 256 GB	$2765	$4.1475	256 GB	56	5000 GB	11 TB	40 Gbps / 11 Gbps
Premium 512 GB	$5530	$8.295	512 GB	64	7200 GB	12 TB	40 Gbps / 12 Gbps

Figure 7: Linode Breakdown - Premium 64GB VM used for modelling

there is no doubt in the ability to implement such research to similar industry environments - however, this would be another area of research in itself.

2.13.3 CSSHx ussage

To save myself from repeating the same setup procedure across all machines in my cluster, and to avoid making any human error that would cause one machine to be less efficient than others, I made use of an application called CSSHx that allows a user to connet to multiple SSH Sessions and control them all simultaneously - running the same command across the cluster.

Stocks to be analysed are specifiied in a plain text format, and so splitting the workload across machines in the cluster meant splitting the text file between the machines, and no changes to the analysis pipeline itself.

See Figure 8 overleaf.

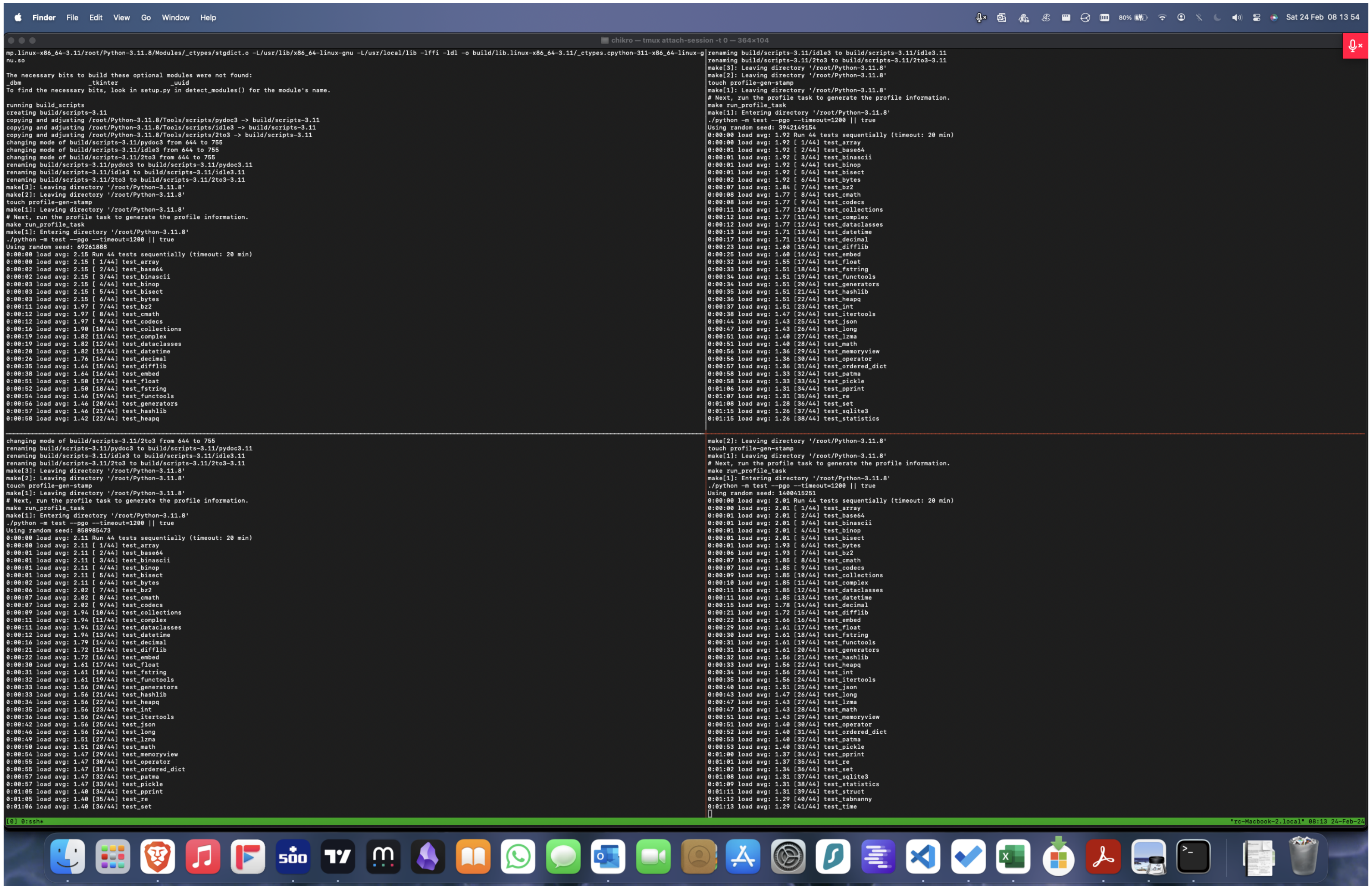


Figure 8: CSSHx Usage - Environment setup of 4 cloud VPCs in parallel

3 The Black-Scholes-Merton model

3.1 The concept

The Black-Scholes-Merton (BSM) model serves as an extension and refinement of the original Black-Scholes (BS) model, enhancing its applicability to a broader range of financial scenarios. The BSM model incorporates the impact of dividend payments by introducing a continuous dividend yield, allowing it to accurately value options on assets that pay dividends during the option's life. The BSM model also accommodates variations in the risk-free interest rate over time, providing a more realistic representation of market conditions. The Black-Scholes-Merton model builds upon the success of the Black-Scholes framework, making it a versatile and widely used tool for option pricing, especially in the presence of dividend-paying stocks.

The BSM model defines a set of partial differential equations that solve for options price, and can be differentiated to find certain properties of Options, known as "Greeks". For pricing options, we can simply use the closed form representations. The closed-form solutions provide analytical expressions that directly calculate the option prices without the need for numerical methods.

Call Option:

The price (C) of a European call option is given by:

$$C = S_0 \cdot N(d_1) - X \cdot e^{-rT} \cdot N(d_2)$$

Put Option:

The price (P) of a European put option is given by:

$$P = X \cdot e^{-rT} \cdot N(-d_2) - S_0 \cdot N(-d_1)$$

Where:

S_0 is the current stock price
X is the option's strike price
r is the risk-free interest rate
T is the time to maturity (in years)
N is the cumulative distribution function of the standard normal distribution
d_1 and d_2 are calculated as follows:

$$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2) \cdot T}{\sigma \cdot \sqrt{T}}$$

$$d_2 = d_1 - \sigma \cdot \sqrt{T}$$

σ is the volatility of the stock's returns

[5]

3.2 Implementation

We produce an implementation in python directly for the above mentioned definitions of the BSM model.

```
def bsm_option_price(S, K, T, r, sigma, option_type, q=0):
    # Calculate d1 and d2 using Black-Scholes-Merton formulas
    d1 = (np.log(S / K) + (r - q + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)

    # Calculate option price based on option type
    if option_type == 'C':
        option_price = S * np.exp(-q * T) * norm.cdf(d1) - K * np.exp(-r * T) *
    norm.cdf(d2)
    elif option_type == 'P':
        option_price = K * np.exp(-r * T) * norm.cdf(-d2) - S * np.exp(-q * T) *
    norm.cdf(-d1)
    else:
        raise ValueError()

    return option_price
```

We then run this on every row we've extracted from Yahoo Finance. Due to the comprehensive setup of the aforementioned Testing Framework, implementing models for accuracy analysis is incredibly straight forward.

3.3 The derivation

We also look at, in further depth, how the BSM model is derived, as it will aid in future derivations of custom models.

Wiener Process

A *Markov Process* is stochastic (well-described by a random probability distribution) process where only the present value of a random variable is relevant for the future outcome. This is sometimes also known as the *Memoryless Property* or the *Markov Property*. [28] A **Wiener Process** is a type of Markov Stochastic process with mean 0 and variance 1 per annum. [16]

The Wiener process, also known as Brownian motion (used in simulating random walks), is a continuous-time stochastic process $W(t)$ with the following properties:

1. $W(0) = 0$ almost surely.
2. The increments $W(t_2) - W(t_1)$ for $t_2 > t_1$ are independent Gaussian random variables.
3. The increments have mean zero and variance $t_2 - t_1$.
4. It has continuous sample paths.

Ito Process

An Ito process is a generalised Wiener Process, named after the Japanese mathematician Kiyoshi Itô. It is a stochastic process $X(t)$ that can be represented as the sum of a deterministic component and a stochastic component, where the stochastic component is given by a Wiener process. Mathematically, an Ito process is expressed as:

$$dX(t) = \mu(t)dt + \sigma(t)dW(t) \tag{1}$$

where:

- $dX(t)$ represents the infinitesimal change in X at time t.
- $\mu(t)$ is the drift coefficient, representing the deterministic component.
- $\sigma(t)$ is the diffusion coefficient, representing the stochastic component.
- $dW(t)$ represents the increment of the Wiener process at time t.

This stochastic differential equation (SDE) captures the dynamics of many processes encountered in finance, physics, and other fields.

Reference for this section can be found in Hull [16], Section 13.3.

Stochastic Differential Equations

Using what we now know about Ito Processes, we can consider the stock price S modeled as an Ito process itself:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where:

- μ represents the drift (expected return),
- σ is the volatility of the stock returns,
- dW_t denotes the increment of a Wiener process, encapsulating the random market fluctuations.

Reference for this section can be found in Hull [16], Section 13.3.

Risk-Neutral Valuation

In risk-neutral valuation as specified by the Fundamental Theorem of Asset Pricing, we assume $\mu = r$, the risk-free rate. This modifies the SDE to:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

This substitution is justified by the need to discount future cash flows at the risk-free rate, as discussed before and in more detail in Hull [16], Section 13.8.

Ito's Lemma Formula

Given $f(S_t, t)$, a twice differentiable function in S and once in t, Ito's Lemma states that:

$$df(S_t, t) = \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S_t \frac{\partial f}{\partial S} dW_t.$$

This result is critical as it allows us to express changes in the option price due to infinitesimal movements in the stock price and time (Hull, Section 13.7).

Reference for this section from Hull [16], Section 13.7.

3.3.1 Derivation of Closed-Form Solution

Using the risk-neutral process and applying Ito's Lemma to the option pricing function, we derive the Black-Scholes partial differential equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

Solving this PDE for boundary conditions specific to European options, we use the terminal payoff conditions:

$$C_T = \max(S_T - K, 0) \quad \text{and} \quad P_T = \max(K - S_T, 0).$$

The solution to the PDE with these boundary conditions leads to the Black-Scholes formulas for European call and put options:

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2),$$

$$P = Ke^{-rT} N(-d_2) - S_0 N(-d_1),$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

and $N(x)$ is the cumulative distribution function of the standard normal distribution.

These formulas were derived by solving the PDE using the method of change of variables and then transforming back to the original variables, ensuring alignment with financial theory and market observations Hull [16], Section 15.4.

Most importantly, the use of Ito's Lemma, Wiener Processes, and Stochastic Differential Equations are prominent in many derivations of options pricing models, and naturally will also be included in the one that we attempt to derive.

3.4 General Evaluation

Metric	Value
Mean Absolute Difference (MAD)	7.65
Root Mean Squared Error (RMSE)	7.66
Mean Percent Change Difference (MPCD)	6.66%
Correlation Coefficient	0.716

Table 1: Aggregated Performance Metrics of the BSM Model

The aggregated performance metrics present a detailed view of the model's accuracy:

- **Mean Absolute Difference (MAD):** The model produces an average MAD of 7.65, which quantifies the average magnitude of errors in the model's predictions without considering their direction. This metric is essential for understanding the typical error size that traders and analysts might expect when using the BSM model for pricing options in different sectors.
- **Root Mean Squared Error (RMSE):** At an average of 7.66, the RMSE provides insight into the error variance, offering a measure of the model's accuracy by highlighting the square root of the average of squared differences between predicted and actual prices. This is particularly useful for identifying how significantly the model's predictions deviate from true market values, with a higher RMSE indicating less reliability.
- **Mean Percent Change Difference (MPCD):** The MPCD averages 6.66%, reflecting the typical percentage deviation between the model's predictions and the actual market prices. This percentage is critical for investors and financial analysts who evaluate the relative error in terms of percentage changes.
- **Correlation Coefficient:** The model's average Correlation Coefficient of 0.716 signifies a strong linear relationship between the predicted and actual prices across various assets. This high correlation indicates that, generally, the BSM model can effectively capture the direction of price movements, though its precision may vary.

In general, the BSM model captures fundamental behavior of stocks, but lacks accuracy. We now look at subset of data to see if the model excels under specific circumstances.

...continued overleaf

3.5 Sector Performance

Sector	RMSE	MPDC	Correlation
Aerospace	0.596	0.030	0.990
Consumer Goods	1.924	0.051	0.655
Energy	2.938	0.069	0.623
Financial	4.237	0.037	0.690
Technology	9.575	0.052	0.662

Table 2: BSM Metrics by Sector

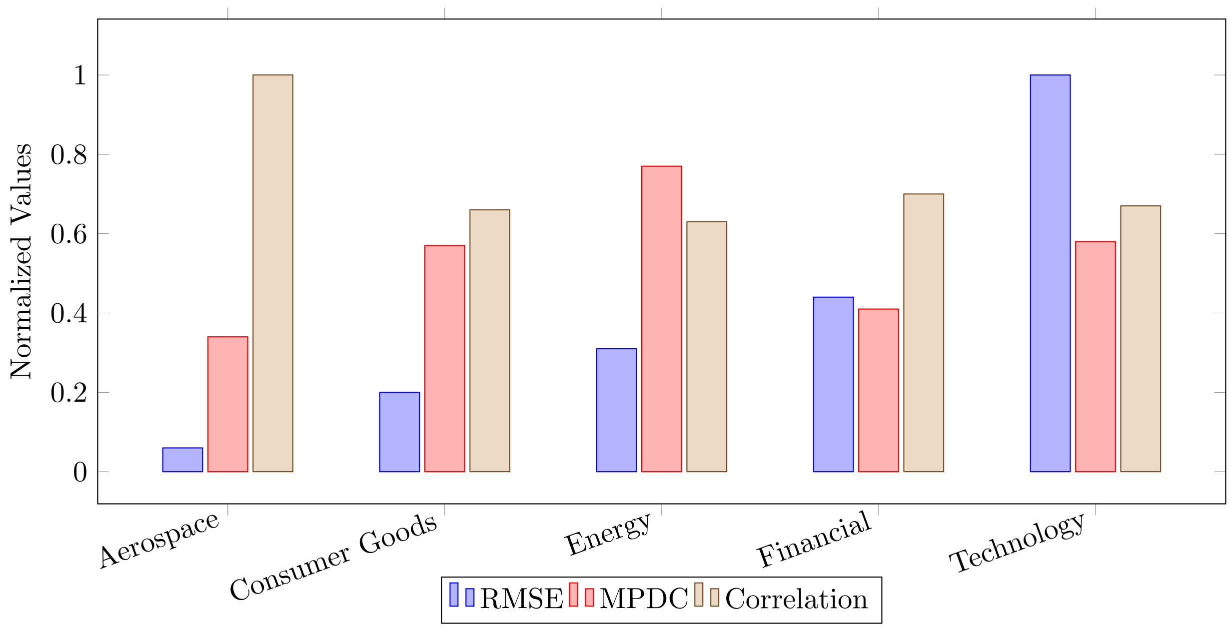


Figure 9: Normalised Black Scholes Model Metrics by Sector

The evaluation of the Black-Scholes-Merton (BSM) model's performance across various sectors reveals detailed insights into its accuracy in modelling market prices amidst sector-specific dynamics.

3.5.1 Technology Sector

Apple Inc. (AAPL), *Microsoft Corporation (MSFT)*, and *Nvidia Corporation (NVDA)* etc.

Within the technology sector, the BSM model highlights notable challenges it faces as a result of rapid technological advancements and market disruptions, this sector poses unique obstacles for price prediction models. The BSM model showcases the highest error metrics in this domain, coupled with a comparatively lower correlation coefficient of *0.662*. Such findings underscore the inherent difficulties in accurately modeling stock prices within highly volatile and innovation-driven markets.

3.5.2 Aerospace Sector

Boeing Co. (BA) and *Lockheed Martin Corp. (LMT)* etc.

Contrary to the technology sector, the aerospace sector demonstrates impressive performance metrics when evaluated. Here, precision and correlation both emerge as distinguishing features, with a high correlation coefficient of *0.990*, indicating the model's robustness in environments characterized by stability and predictability. The aerospace industry's relatively steady growth trajectories and long-term investment horizons align favorably with the assumptions underlying the BSM framework, resulting in high accuracy and correlation.

3.5.3 Energy Sector

Exxon Mobil Corp. (XOM) and *Chevron Corp. (CVX)* etc.

The energy sector presents, once again, a challenging landscape for the BSM model, marked by heightened error metrics and the lowest correlation coefficient among the sectors analyzed, at *0.623*. Factors such as fluctuating oil prices, geopolitical tensions, and macroeconomic indicators exert significant influence on stock prices within this domain. The BSM model's limitations in accounting for such external variables contribute to its diminished performance in accurately forecasting market prices within the energy sector.

3.5.4 Financial Sector

JPMorgan Chase & Co. (JPM) and *Goldman Sachs Group Inc. (GS)* etc.

In the financial sector, the BSM model demonstrates moderate performance metrics with a correlation coefficient of *0.690*. Here, factors such as economic indicators, central bank policy changes, and regulatory frameworks exert considerable influence on asset prices. While the BSM model offers insights into option pricing within this domain, its predictive capabilities are dampened by the complexities of financial markets.

3.5.5 Consumer Goods Sector

Procter & Gamble Co. (PG) and *The Coca-Cola Company (KO)* etc.

The consumer goods sector again showcases a mixed performance when evaluated. While the model exhibits reasonable accuracy in predicting price movements, the variability in MPCD remains pronounced, with a correlation coefficient of *0.655*. This variability can be attributed to external market forces, including macroeconomic fluctuations and structural shifts in the economy. Such factors, which extend beyond the sight of the BSM model's parameters, contribute to the observed discrepancies in forecasting accuracy within the consumer goods sector.

3.5.6 Variability Across Sectors

The variability in these performance metrics across different sectors indicates that the BSM model's applicability and accuracy are not uniform. For instance:

- **High Correlation in Stable Sectors:** Sectors with stable market conditions, such as aerospace, where Boeing (BA) operates, exhibit exceptionally high correlation coefficients (approximately 0.992). This high correlation underscores the model's robustness and reliability in environments with less volatility or fewer market disruptions.

- **Performance in Volatile Sectors:** Conversely, in sectors characterized by high volatility and dynamic market conditions, the model may exhibit lower correlation coefficients and higher RMSE, suggesting that the BSM model's assumptions—such as constant volatility and log-normal distribution of stock prices—may not hold well. These sectors require a more nuanced application of the model, potentially incorporating adjustments for stochastic volatility or leveraging alternative models that better accommodate irregular price movements.

This sector-based performance analysis reveals that while the Black-Scholes-Merton model remains a powerful tool for financial modeling, its effectiveness can vary significantly depending on the underlying sector characteristics. This necessitates a tailored approach to its application in financial analysis, taking into consideration the specific volatility patterns and market behaviors of each sector. It also outlines the difficulty of modelling sectors with external forces influencing the market, which models cannot simulate from looking at just the underlying asset of the option contract in consideration.

3.6 High Volume Asset Performance

High Volume Assets are those that are most highly traded across markets.

The performance of the Black-Scholes-Merton (BSM) model on high-volume assets provides further insights, this section examines assets with significant trading volumes and varying levels of market activity, providing a comparative analysis of the BSM model's performance metrics.

Performance Metrics Table: The following table provides a detailed breakdown of the BSM model performance across selected high-volume assets, showcasing the diversity in model accuracy:

Ticker	BSM RMSE	BSM Correlation Coefficient	BSM MPCD
AAPL	4.50	0.579	5.48%
AMD	4.77	0.899	3.29%
AMZN	2.97	0.601	1.96%
META	4.80	0.972	4.08%
MSFT	17.08	0.753	5.16%
NVDA	19.46	0.981	3.02%
TSLA	6.59	0.862	10.21%

Table 3: Performance metrics of the BSM model on selected high volume assets

Differential RMSE and Correlation Coefficients: The RMSE (Root Mean Squared Error) values and correlation coefficients show significant variability among high-volume assets, suggesting that the BSM model's efficacy is not uniform across different market conditions and asset classes.

- **Stable Sectors:** Companies such as Amazon (AMZN) and Microsoft (MSFT), which operate in relatively stable sectors like e-commerce and software, demonstrate varied RMSE values. Amazon shows lower RMSE and moderate correlation, indicating a reasonable prediction accuracy, while Microsoft, with higher RMSE, shows the model's challenges in these sectors.
- **Volatile Sectors:** Tech companies such as Nvidia (NVDA), Meta Platforms (META), and Tesla (TSLA) exhibit much higher RMSE values. Despite these higher errors, their correlation coefficients are generally high (NVDA at 0.981, META at 0.972, and TSLA at 0.862),

suggesting that the BSM model, while capturing the direction of price movements well, struggles with the magnitude, especially in high-volatility environments like the technology and automotive sectors.

3.7 Technical Challenges in Volatile Markets

We've seen from the data that the BSM Model consistently struggles with high volatility markets, both when we look at specific sectors and also when considering heavily traded securities, so let's take a look on a technical level the reasons for this.

The BSM model, while foundational in financial modeling, bases its calculations on several critical assumptions (as introduced earlier) that can be restrictive in certain market environments, particularly in sectors characterized by high volatility such as technology. Here we delve deeper into the mathematical implications of these assumptions and their impact on the model's effectiveness in volatile markets.

3.7.1 Assumption of Constant Volatility

One of the central assumptions of the BSM model is that volatility (σ), which measures the rate and magnitude of changes in prices, remains constant over the option's life. This assumption simplifies the computation of the model by using a single volatility figure derived from historical price data of the underlying asset. [18]

Mathematical Implication: The BSM model assumes volatility using a logarithmic function as part of the formula to calculate the option's price:

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where S is the stock price, K is the strike price, r is the risk-free rate, T is the time to expiration, and σ is the volatility. The terms d_1 and d_2 are inputs to the cumulative distribution function of the standard normal distribution, which then calculates probabilities essential in determining the expected monetary value of holding the option.

Issue in High Volatility Markets: In sectors with hightened volatility, such as the technology sector, where companies such as Nvidia and Meta operate, the assumption of constant volatility quickly becomes very problematic. Rapid information flow and frequently changing market events lead to significant fluctuations in volatility. When volatility is modelled as constant, it leads to heavy underestimations or overestimations of risk in option pricing, and hence manifesting as significant pricing errors.

3.7.2 Assumption of Log-Normal Distribution

Another fundamental assumption of the model, is that prices of underlying assets follow a perfectly log-normal distribution. This means that it assumes the logarithm of asset prices is normally distributed - which ensures that the asset prices themselves remain positive and that the pricing distribution is skewed. This, very generally speaking, meas that it aligns with observed stock prices.

Mathematical Implication: The log-normal distribution is used to model the future prices of the stocks under the risk-neutral measure, facilitating the use of the Black-Scholes formula to find the expected payoff of the option. The probability density function of a log-normally distributed variable X is given by:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log x - \mu)^2}{2\sigma^2}\right)$$

where μ and σ are the mean and standard deviation of the variable's natural logarithm.

Issue in High Volatility Markets: In volatile markets, the rapid price changes can lead to "fat tails" where extreme values are more common than what the log-normal distribution predicts. This discrepancy can result in mispricing, particularly for options that are deep out-of-the-money or in-the-money, where the sensitivity to changes in the tail of the distribution is greater.

3.8 Implications for Financial Practices

The variability in the BSM model's performance across different sectors underscores the importance of sector-specific approaches in financial modeling. This means that investors should always consider the underlying market conditions and asset characteristics when employing the BSM model for options pricing, risk management, or strategic planning - naturally, this takes away from the optimistically automated method we would hope from an options pricing model.

3.9 Potential Model Enhancements

To address these challenges, I considered several enhancements:

- **Incorporating Stochastic Volatility Models:** Integrating models that account for changes in volatility over time could improve prediction accuracy, particularly for assets in high-volatility sectors.
- **Adaptive Techniques:** Utilizing machine learning algorithms to adaptively tune model parameters based on real-time data could reduce the lag in response to market conditions.
- **Hybrid Models:** Combining the BSM framework with other financial models that account for jumps or discrete changes in asset prices could provide a more generalistic approach to options pricing.

We first investigate another foundational options pricing model, the Binomial Option Pricing Model, before discussing in further detail how some of these enhancements could be implemented.

4 The Binomial Options Pricing model

4.1 The concept

The Binomial Tree Model is a discrete-time model used to approximate the evolution of the stock price over time. The model assumes that at each step, the stock price S can move up by a factor of u or down by a factor of d with corresponding probabilities p and $1 - p$.

1. The stock price at time t is S_t.
2. At time $t + \Delta t$, the stock price can be either uS_t or dS_t.
3. The up and down factors, u and d, are constants determined by the volatility of the stock and the length of the time step Δt.
4. p is the risk-neutral probability of the stock price moving up.

4.2 Implementation

We produce an implementation in python directly for the above mentioned definitions of the BOPM model.

```python
def binomial_option_pricing(stock_price, strike_price, time_to_maturity, risk_free_rate, volatility, dividend_yield, option_type, steps=20):
    # Time step
    dt = time_to_maturity / steps
    # Up and down factors
    u = np.exp(volatility * np.sqrt(dt))
    d = 1 / u
    # Risk-neutral probability
    p = (np.exp((risk_free_rate - dividend_yield) * dt) - d) / (u - d)

    # Initialize stock price tree
    stock_prices = np.zeros((steps + 1, steps + 1))
    for i in range(steps + 1):
        for j in range(i + 1):
            stock_prices[j, i] = stock_price * (u ** j) * (d ** (i - j))

    # Initialize option value tree
    option_values = np.zeros_like(stock_prices)

    # Option value at final nodes
    if option_type == 'c':  # Call
        option_values[:, steps] = np.maximum(stock_prices[:, steps] - strike_price, 0)
    elif option_type == 'p':  # Put
        option_values[:, steps] = np.maximum(strike_price - stock_prices[:, steps], 0)

    # Backward calculation for option price
    for i in range(steps - 1, -1, -1):
        for j in range(i + 1):
            early_exercise = 0
            if option_type == 'C':
                early_exercise = max(stock_prices[j, i] - strike_price, 0)
            elif option_type == 'P':
                early_exercise = max(strike_price - stock_prices[j, i], 0)
```

```
                option_values[j, i] = max(early_exercise, np.exp(-risk_free_rate * dt) * (p * option_values[j, i + 1] + (1 - p) * option_values[j + 1, i + 1]))

    return option_values[0, 0]
```

We then run this on every row we've extracted from Yahoo Finance. Due to the comprehensive setup of the aforementioned Testing Framework Setup, implementing models for accuracy analysis is incredibly straight forward.

4.3 The Derivation

We delve further into the derivation of the Binomial Options Pricing Model (BOPM), which forms the foundation for understanding more complex derivatives and models.

Single-Step Binomial Model

Consider a single-step binomial model:

$$S_{t+\Delta t} = \begin{cases} uS_t & \text{with probability } p \\ dS_t & \text{with probability } 1-p \end{cases}$$

The expected stock price under the risk-neutral measure is:

$$E[S_{t+\Delta t}] = puS_t + (1-p)dS_t = e^{r\Delta t}S_t$$

where r is the risk-free rate, ensuring the stock's expected return equals the risk-free rate over Δt. [8]

This follows the same idea as when we assumed the risk neutral measure in the BSM model.

Recursive Pricing Formula

To extend onwards from a single step model, we then use the recursive nature of the binomial model, the price of the derivative at time t, V_t, can be computed as the discounted expected value of its payoff at $t + \Delta t$:

$$V_t = e^{-r\Delta t}\left(pV^u_{t+\Delta t} + (1-p)V^d_{t+\Delta t}\right)$$

where $V^u_{t+\Delta t}$ and $V^d_{t+\Delta t}$ are the values of the derivative if the stock price moves up or down, respectively. [3]

Recall the discounted expected pay-off is to keep in line with the Fundamental Theorem of Asset pricing, where we always discount future predicted values to current values by accounting for the risk-free rate and the risk-neutral probabilities defined above.

Convergence to Black-Scholes

As the number of steps in the binomial tree increases and the time step decreases, the binomial model converges to the continuous-time model described by the Black-Scholes equation. This convergence is can be proven using the Central Limit Theorem, it is an immediate result of the sum of the binomial increments converging to a lognormal distribution; the underlying assumption in the Black-Scholes and Black-Scholes-Merton model. [8]

Closed-Form Solution

For European options, where exercise is only permitted at maturity, the binomial model provides a closed-form solution resembling the Black-Scholes formula under certain parameter configurations (specifically as the number of steps goes to infinity). The binomial model's flexibility also allows for the valuation of American options, where early exercise is permitted, showing its advantage over the Black-Scholes model in handling early exercise features. We account for American options by discounting the predicted values of the option at all stages, as opposed to just at the end of the tree in the European case, and so we use a recursive implementation.

References for this section can be found in Cox, Ross, and Rubinstein (1979), who introduced the binomial model to value complex derivatives based on the no-arbitrage condition and risk-neutral valuation. [8]

4.4 General Evaluation

Metric	Value
Mean Absolute Difference (MAD)	8.07
Root Mean Squared Error (RMSE)	8.09
Mean Percent Change Difference (MPCD)	5.68%
Correlation Coefficient	0.749

Table 4: Aggregated Performance Metrics of the BOPM Model

The aggregated performance metrics present a detailed view of the model's accuracy:

- **Mean Absolute Difference (MAD):** The BOPM produces an average MAD of 8.07, slightly higher than the BSM's MAD of 7.65, indicating a slightly larger typical error size compared to the BSM model.
- **Root Mean Squared Error (RMSE):** At an average of 8.09, the RMSE for the BOPM is also slightly higher than the BSM's 7.66. This metric provides insight into the error variance, the higher RMSE for BOPM suggests it may be less reliable than the BSM in terms of predicting market values accurately, and is expected as a result of the increased MAD result.
- **Mean Percent Change Difference (MPCD):** The MPCD for the BOPM averages 5.68%, which is lower than the BSM's 6.66%. This reflects a typically smaller percentage deviation between the BOPM's predictions and the actual market prices compared to the BSM.
- **Correlation Coefficient:** The BOPM's average Correlation Coefficient of 0.749 is slightly higher than the BSM's 0.716, indicating a stronger linear relationship between the predicted and actual prices for the BOPM. This suggests that, generally, the BOPM might be more effective at capturing the direction of price movements compared to the BSM, though its overall precision may still vary much like the BSM.

Using the BSM as a starting point, these comparisons provide an insightful look into how both models performs under varying market conditions, highlighting the strengths and limitations of each in predicting market prices accurately. However, we again look into subsets of the results to spot any particular trends where BOPM may out perform BSM.

4.5 Sector Performance

Sector	RMSE	MPDC	Correlation
Aerospace	0.707	0.033	0.994
Consumer Goods	2.012	0.053	0.653
Energy	3.069	0.073	0.656
Financial	4.425	0.038	0.742
Technology	10.078	0.048	0.677

Table 5: BOPM Metrics by Sector

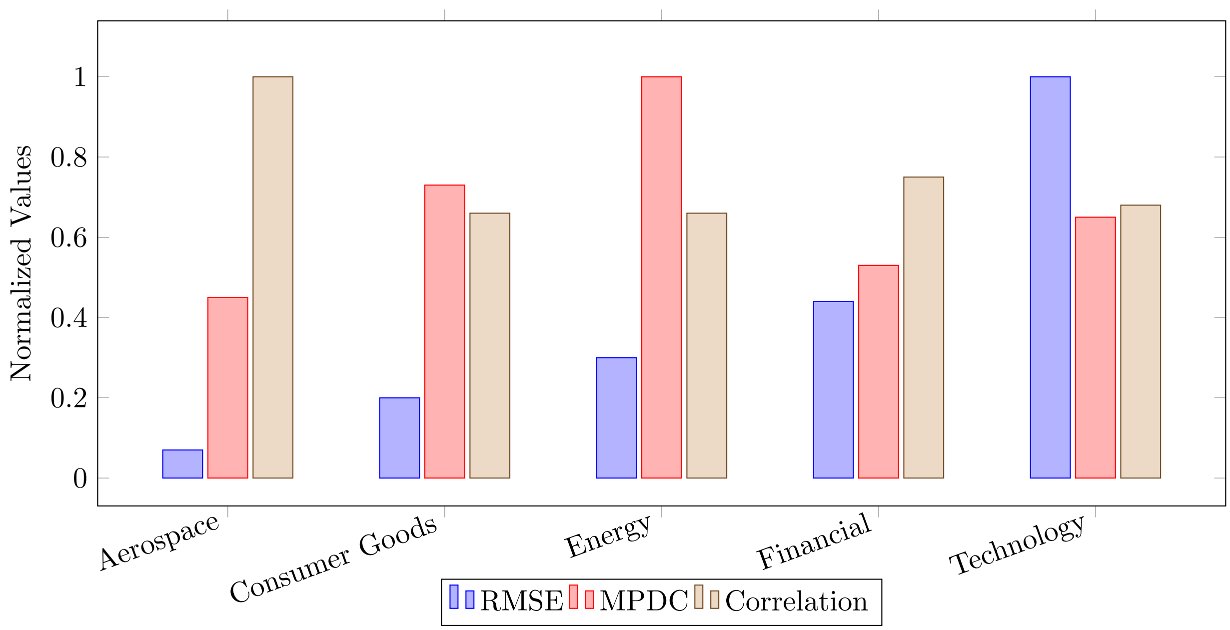


Figure 10: Normalised Binomial Options Pricing Model Metrics by Sector

The evaluation of the Binomial Options Pricing Model (BOPM) performance across diverse sectors reveals nuanced insights into its performance, often showing contrast to the Black-Scholes-Merton (BSM) model.

4.5.1 Technology Sector

Apple Inc. (AAPL), *Microsoft Corporation (MSFT)*, and *Nvidia Corporation (NVDA)* etc.

Within the technology sector, the BOPM model faces significant challenges, exhibiting the highest error metrics and a moderate correlation coefficient of *0.677*, compared to *0.662* in BSM. Showing a neglible improvement over BSM.

4.5.2 Aerospace Sector

Boeing Co. (BA) and *Lockheed Martin Corp. (LMT)* etc.

The aerospace sector shows exemplary performance with the BOPM model, marked by the lowest RMSE and the highest correlation coefficient of *0.994*, significantly better than BSM's *0.990*. This indicates robustness in stable environments, aligning well with the industry's growth trajectories.

4.5.3 Energy Sector

Exxon Mobil Corp. (XOM) and *Chevron Corp. (CVX)* etc.

The energy sector remains challenging for BOPM, with high error metrics and the lowest correlation among sectors at *0.656*, slightly better than BSM's *0.623*. Fluctuating oil prices and geopolitical tensions heavily influence stock prices, revealing limitations in the BOPM's forecasting capabilities, albeit less so than the BSM's.

4.5.4 Financial Sector

JPMorgan Chase & Co. (JPM) and *Goldman Sachs Group Inc. (GS)* etc.

The financial sector exhibits moderate RMSE and the second highest correlation of *0.742* with BOPM, surpassing BSM's *0.690*. However, once again, this sector's complexity, influenced by economic indicators and policy changes, still presents challenges for the BOPM much like the BSM.

4.5.5 Consumer Goods Sector

Procter & Gamble Co. (PG) and *The Coca-Cola Company (KO)* etc.

The consumer goods sector shows reasonable accuracy with a specifically pronounced variability in MPCD with BOPM, with a correlation of *0.653*, marginally lower than BSM's *0.655*. This suggests external market forces continue to play a significant role in affecting the model's predictive accuracy.

4.6 High Volume Asset Performance

High Volume Assets are those that are most highly traded across markets.

The performance of the Binomial Options Pricing Model (BOPM) on high-volume assets provides further insights. This section examines assets with significant trading volumes and varying levels of market activity, providing a comparative analysis of the BOPM model's performance metrics.

Performance Metrics Table: The following table provides a detailed breakdown of the BOPM model performance across selected high-volume assets, showcasing the diversity in model accuracy:

Differential RMSE and Correlation Coefficients: The RMSE (Root Mean Squared Error) values and correlation coefficients for the BOPM model also show significant variability among high-volume assets, similar to observations from the BSM model. This variability suggests that the BOPM model's efficacy, like that of the BSM model, is not uniform across different market conditions and asset classes.

Ticker	BOPM RMSE	BOPM Correlation Coefficient	BOPM MPCD
AAPL	4.89	0.606	4.71%
AMD	5.09	0.924	3.74%
AMZN	3.41	0.733	2.48%
META	5.18	0.973	4.35%
MSFT	17.66	0.745	5.31%
NVDA	20.58	0.985	3.20%
TSLA	6.81	0.894	5.68%

Table 6: Performance metrics of the BOPM model on selected high volume assets

- **Stable Sectors:** Companies such as Amazon (AMZN) and Microsoft (MSFT), which operate in relatively stable sectors like e-commerce and software, demonstrate varied RMSE values with the BOPM model. Amazon shows a lower RMSE and a higher correlation compared to its BSM performance, indicating better prediction accuracy under the BOPM framework. In contrast, Microsoft, with a higher RMSE, reveals ongoing challenges for both models in accurately forecasting stock prices in stable sectors.
- **Volatile Sectors:** Tech companies such as Nvidia (NVDA), Meta Platforms (META), and Tesla (TSLA) exhibit higher RMSE values with the BOPM model, similar to the BSM model. Despite these higher errors, their correlation coefficients are very high (NVDA at 0.985, META at 0.973, and TSLA at 0.894 with BOPM), suggesting that while the BOPM model, like the BSM, captures the direction of price movements well, it struggles with the magnitude, particularly in high-volatility environments such as the technology and automotive sectors. This indicates a consistent limitation across models in dealing with assets subject to rapid market changes and innovations.

4.7 Technical Challenges in Volatile Markets

We've observed from the data that the BOPM model, similar to the BSM model, faces difficulties in high volatility markets, particularly evident when analyzing specific sectors and heavily traded securities. Here we explore the technical reasons underpinning these challenges with the BOPM model.

4.7.1 Discrete Time and Price Movements

Unlike the BSM model which assumes a continuous time and price framework, the BOPM model operates on discrete time steps and price movements. This discretization can introduce inaccuracies in highly volatile markets. [7]

Mathematical Implication: In the BOPM, the evolution of stock prices over time is modeled using a binomial tree, where each node represents a possible future price, which can either move up by a factor u or down by a factor d, calculated as follows:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = \frac{1}{u}$$

where σ is the volatility and Δt is the time step size. The option price is then determined by backward induction, calculating the expected payoff at each node.

Issue in High Volatility Markets: In markets where volatility is high, the assumption that price movements are symmetrically distributed in upward and downward movements becomes less valid. Rapid and significant price shifts can lead to financial situations such that the discrete steps do not sufficiently capture the true path of stock prices, potentially leading to significant mispricing, in particular this is exaggerated for short-term options.

4.7.2 Fixed Probabilities

BOPM assumes fixed probabilities for up and down movements in the stock price, which may not hold in volatile markets where probabilities themselves could be dynamic.

Mathematical Implication: The probability of an upward movement in stock price p in the BOPM is typically defined as:

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

where r is the risk-free rate. This probability is used to weight the expected payoffs in the binomial tree and so to derive the option's price.

Issue in High Volatility Markets: In volatile markets, the likelihood of drastic price changes increases, and the fixed nature of p may not accurately reflect the true sentiment of the market. This static probability does not account for sudden shifts in market sentiment or unexpected news, leading to inaccuracies in pricing particularly noticeable in assets like Tesla or Nvidia, which are subject to rapid, news-driven price changes.

These technical nuances of the BOPM model highlight its limitations in volatile environments. While the model offers the flexibility of adjusting the time steps and the up and down factors, it still may not fully capture the complexities of markets driven by high volatility and dynamic shifts, unlike the BSM model, which, despite its assumptions of continuous markets and log-normal distribution, also struggles but due to different underlying theoretical constraints.

4.8 Potential Model Enhancements

To address these challenges, several enhancements could be considered:

- **Incorporating Stochastic Tree Models:** Integrating approaches that allow for dynamic adjustments in the tree probabilities and volatility estimates could improve prediction accuracy, particularly for assets in high-volatility sectors.
- **Adaptive Techniques:** Employing machine learning algorithms to dynamically adjust binomial tree parameters based on real-time data could mitigate some of the model's limitations related to fixed structural assumptions.
- **Hybrid Models:** Combining the BOPM framework with other financial models that account for more complex market dynamics, such as those incorporating jump diffusion processes, could provide a more comprehensive approach to options pricing.

Having now investigated both the Black-Scholes-Merton and Binomial Options Pricing Model, we take a look at a few of the potential model enhancements and their implementation in more detail below.

5 Parameter improvement

5.1 Implied Volatility

Volatility is the true volatility, that can only be known in hindsight. Implied volatility is the market's perceived expectation of current volatility.

The Black-Scholes model produces valuations relying on the fact that the underlying can be modelled using random walks with constant volatility. If this is true, then the distribution of movements of an asset at any point in time, at any strike price is perfectly log-normal. As well as this, it suggests that all options on the same underlying asset must have the same implied volatility - clearly this is not true.

By changing the way we use parameters and by making the volatility non-constant, we are essentially attributing a non-lognormal distribution on the underlying asset. To value European options, by calculating the expected values of their pay-offs, optimally we need to know the exact form of the non-lognormal distribution. To value American or more exotic options, you must know the exact nature of the modified random walk – that is, how the volatility varies stochastically with stock price and time. This extension of modelling implied volatility is often known as the *Implied Volatility Smile* [10]

5.1.1 Implied Volatility Smile

Diverging from the assumptions of the BSM model, volatilities observed are higher for options that are far from being at the money, particularly for puts following market downturns, indicating a negative skew for options on equity markets.

To address this, the modified random walk model is introduced where both the drift $\mu(t)$ and volatility $\sigma(S,t)$ of the stock price process S are functions of the stock price S and time t. The modified stochastic differential equation (SDE) is expressed as:

$$dS = \mu(S,t)\,dt + \sigma(S,t)\,dZ \tag{2}$$

where dZ represents the increment of a Wiener process. This replaces the constant volatility σ in the Black-Scholes model with a local volatility $\sigma(S,t)$ that adapts to the volatility smile reflected in market data. [10]

5.1.2 Formulaic Representation and Computation

Similar to the derivation of the BSM model, we can recompute the stock price equation using a stochastic model for implied volatility, as follows.

Local Volatility Model

Incorporating the volatility smile, the stock price's stochastic differential equation (SDE) includes a local volatility function, which is a function of both the stock price S and time t:

$$dS_t = \mu(S_t,t)S_t\,dt + \sigma(S_t,t)S_t\,dW_t \tag{3}$$

where $\sigma(S_t,t)$ reflects the structure of the implied volatility smile, capturing changes in volatility with the price level and over time, and most importantly - at every time step.

Applying Itô's Lemma

Transform $X_t = \log S_t$ and apply Itô's Lemma to derive:

$$dX_t = \left(\mu(S_t, t) - \frac{1}{2}\sigma^2(S_t, t)\right) dt + \sigma(S_t, t)\, dW_t \tag{4}$$

Integration Over Δt

Assuming that the local volatility and drift are stable over short intervals, integrate the differential to get:

$$X_{t+\Delta t} = X_t + \left(\mu(S_t, t) - \frac{1}{2}\sigma^2(S_t, t)\right) \Delta t + \sigma(S_t, t)(W_{t+\Delta t} - W_t) \tag{5}$$

Final Stock Price Equation

Converting the logarithmic change back to the stock price, assuming $W_{t+\Delta t} - W_t \sim \sqrt{\Delta t} Z$ (with Z standard normal):

$$S_{t+\Delta t} = S_t e^{(\mu(S_t,t) - \frac{1}{2}\sigma^2(S_t,t))\Delta t + \sigma(S_t,t)\sqrt{\Delta t} Z} \tag{6}$$

Now, we have a rudimentary model where the stock price evolution under a risk-neutral measure uses a volatility function derived from the implied volatility smile, with the intention of enhancing the model's realism and accuracy in reflecting market conditions.

5.2 Dividend Yield

Dividend yield is yet another critical factor in option pricing models, as it represents the expected income from an investment in stocks apart from the price changes. In the context of the Black-Scholes model and its derivatives, the dividend yield affects the underlying stock's price decrement, which in turn impacts the pricing of derivatives like options.

To improve model accuracy, incorporating a variable dividend yield can be particularly beneficial. This involves adjusting the dividend yield assumption to reflect more realistic, dynamic investment environments, where dividends can fluctuate based on company performance and broader economic conditions.

5.2.1 Incorporating Dividend Yield into Models

[20] In traditional Black-Scholes frameworks, a constant dividend yield δ is subtracted from the risk-free rate r in the calculation of expected returns under the risk-neutral measure. However, by allowing dividend yield to vary with time and possibly the stock price, we introduce a more flexible and realistic modeling framework. The modified stock price stochastic differential equation (SDE) is then given by:

$$dS = (r - \delta(S, t))S\, dt + \sigma(S, t)S\, dZ \tag{7}$$

where $\delta(S, t)$ is the dividend yield function, which can be modeled based on historical dividend payment data or forecasted dividends.

5.2.2 Impact of Variable Dividend Yield

Variable dividend yield introduces a level of realism into the model by accounting for:

- **Economic Cycles:** During different phases of the economic cycle, companies might increase or decrease their dividend payouts, affecting their stock's total yield.
- **Company Performance:** Individual company performance might lead to increased or decreased dividends, which should be reflected in the pricing of options on that company's stock.

These are particularly important, as we've consistently noticed that the option pricing models we've investigated perform poorly in sectors that are disproportionately effected by external macroeconomic and structural changes.

5.2.3 Formulaic Representation and Computation

Again, using a derivation similar to that of the Local Volatility model, we can gain an equation that incorporates a dividends as a stochastic input parameter, as opposed to a static one.

Variable Dividend Yield Model

Implementing a variable dividend yield, the equation modifies as follows:

$$dS_t = (r - \delta(S_t, t))S_t\, dt + \sigma(S_t, t)S_t\, dW_t \tag{8}$$

Applying Itô's Lemma to Log Prices with Variable Dividends

Using the transformation $X_t = \log S_t$ and applying Itô's Lemma:

$$dX_t = \left(r - \delta(S_t, t) - \frac{1}{2}\sigma^2(S_t, t)\right) dt + \sigma(S_t, t)\, dW_t \tag{9}$$

Integration Over Δt

Over a short interval Δt, integrating the differential yields:

$$X_{t+\Delta t} = X_t + \left(r - \delta(S_t, t) - \frac{1}{2}\sigma^2(S_t, t)\right) \Delta t + \sigma(S_t, t)\sqrt{\Delta t} Z \tag{10}$$

Final Stock Price Equation with Variable Dividends

The final stock price under a variable dividend model becomes:

$$S_{t+\Delta t} = S_t e^{(r-\delta(S_t,t)-\frac{1}{2}\sigma^2(S_t,t))\Delta t+\sigma(S_t,t)\sqrt{\Delta t} Z} \tag{11}$$

This more dynamic approach helps in closer alignment of the model with market behaviors, reflecting the impact of changing dividend policies on option pricing and advanced investment strategies.

6 The Hybrid Multinomial Pricing model

6.1 The concept

The **multinomial options pricing model** extends the binomial options pricing model by allowing for more than two possible outcomes for the price of the underlying asset at each time step. This model provides a more flexible and accurate approach to pricing options, especially when dealing with complex derivatives and volatile markets. The multinomial options pricing model is already theorised by Cheng Few Lee and Jack C. Lee [19]. While their work derives an extension of the binomial options pricing model by allowing for extra branches of the tree at every time step, I aim to take parameter improvements that have been discussed in the previous chapter and implement a model with their inclusion. Namely, using elements of the implied volatility smile in calculating up/down probabilities dynamically. I chose to use the Implied Volatility Smile as inspiration, as the traditional MOPM model has fundamentals founded in the Black-Scholes model, which has high sensitivity to volatility. By modelling the movement of option prices similarly to the way volatility is modelled, I seek to achieve better performance than just the BSM or BOPM model alone.

The table below is representative of the hybrid multinomial option pricing model's tree at every time step

Jump Amount ($)	**Probability**	**Asset Price**
-10	p_1	S_1
$-9\frac{7}{8}$	p_2	S_2
$-9\frac{6}{8}$	p_{n-2}	S_3
...	...	...
$-\frac{1}{8}$	p_{23}	S_{23}
0	p_{24}	S_{24}
$\frac{1}{8}$	p_{25}	S_{25}
...	...	...
$9\frac{6}{8}$	p_{n-2}	S_{n-2}
$9\frac{7}{8}$	p_{n-1}	S_{n-1}
10	p_n	S_n

Table 7: Nature of Jump Probabilities for Asset Prices

6.2 Derivation and Key Features

This section describes the implementation of a hybrid multinomial option pricing model, which diverges from traditional Multinomial Option Pricing Model (MOPM) frameworks by incorporating concepts from the Implied Volatility Smile as discussed under parameter improvements.

6.2.1 Use of Normal Distribution for Movements

- **Traditional MOPM:** Typically defines upward and downward movements with fixed, pre-determined probabilities, similar to the BOPM model we've seen.
- **Implementation:** Uses quantiles of the standard normal distribution to determine movements, where greater movements become less likely compared to smaller movements (but

never impossible):

$$z\_scores = \text{norm.ppf}(quantiles)$$

This modification, inspired by the model of implied volatility, utilizes the volatility smile to adjust the magnitude of price movements according to a distribution of asset returns, departing from the assumption of a strictly log-normal distribution.

6.2.2 Dynamic Calculation of Price Movements

- **Traditional MOPM:** Employs a static set of potential outcomes for each time step.
- **Implementation:** Calculates price movements dynamically using:

$$up\_movements = \exp\left(\text{drift} + \text{volatility} \times z\_scores \times \sqrt{dt}\right), \quad down\_movements = \frac{1}{up\_movements}$$

Here, the drift is computed as $(r - \text{dividend yield} - 0.5 \times \text{volatility}^2) \times dt$. This method adapts to changes in implied volatility, accounting for non-constant volatility which is a fundamental difference from the assumption of constant volatility used in traditional models.

6.2.3 Complex Lattice Construction

This subsection explains the intricate lattice construction in the hybrid Multinomial Option Pricing Model, designed to capture a more realistic distribution of financial asset returns, particularly under conditions influenced by the Implied Volatility Smile.

Traditional MOPM Lattice Structure In traditional models, the lattice represents possible future stock prices using a relatively simple branching pattern. Typically, each node leads to two possible outcomes: an upward or a downward movement, with these movements often assumed to be symmetric and based on fixed probabilities.

Hybrid Multinomial Model Lattice Design

- **Branching Complexity:** Unlike the traditional approach, the enhanced model uses a more complex lattice with multiple branches at each node. This complexity is necessary to model the nuanced movements implied by the volatility smile, where volatility is not constant and depends on both the stock price and time.
- **Distribution of Movements:** The movements are derived from the quantiles of the normal distribution, allowing for a varied range of upward and downward movements:

$$up\_movements = \exp\left(\text{drift} + \text{volatility} \times z\_scores \times \sqrt{dt}\right), \quad down\_movements = \frac{1}{up\_movements}$$

Here, $z\_scores = \text{norm.ppf}(quantiles)$ are calculated to match the desired distribution of returns, effectively integrating the skew and kurtosis seen in real market data.

- **Lattice Construction:** The lattice is constructed dynamically, with each node at each timestep leading to a potentially large number of subsequent nodes:

$$lattice[step, mid\_index + branch] = \text{payoff calculation using final stock price}$$

This calculation reflects the combined effects of multiple potential price movements, each weighted according to the probability derived from their respective quantiles in the distribution.

- **Path Dependency and Calibration:** Each branch's movement factor is not merely a function of the current price but also of the historical path, making the model sensitive to the path taken by the asset.

Modeling Market Realities

- The enhanced lattice model is designed to better capture extreme market movements and tail risks, which, as we've seen, are often underestimated in traditional models. By using a distribution-based approach to define node transitions at every time step, the model incorporates a more realistic portrayal of market conditions, including fat tails and volatility skew, which are characteristic outlier that exist in real-world financial markets.
- This sophisticated approach allows for a more detailed and accurate valuation of options, particularly those sensitive to extreme movements and varying volatility, thus providing a hypothetically superior tool for risk management and financial analysis.

6.2.4 Risk-Neutral Valuation Adjusted for Continuous Returns

- **Traditional MOPM:** Might not explicitly adjust for continuous distributions in node probability calculations.
- **Implementation:** Adheres to risk-neutral valuation principles but enhances the modeling of returns to integrate both drift and stochastic components influenced by the risk-free rate, dividend yield, and local volatility:

$$discount\_factor = \exp(-r \times \frac{\text{time to maturity}}{\text{num steps}})$$

However, the complexity of the MOPM model can be a drawback as it demands significant computational resources and sophisticated assumptions about the probabilities and outcomes. The increased number of parameters also introduces more potential for estimation errors. Nevertheless, since computational efficiency is not the aim of this project, we overlook this issue.

6.3 Implementation

We produce an implementation in python directly for the above mentioned definitions of the MOPM model.

```python
def normal_distributed_lattice_option_pricing(stock_price, strike_price,
    time_to_maturity, risk_free_rate, volatility, dividend_yield, option_type,
    num_branches=20, num_steps=20):
    dt = time_to_maturity / num_steps
    r = risk_free_rate
    drift = (r - dividend_yield - 0.5 * volatility ** 2) * dt
    sqrt_dt = np.sqrt(dt)

    quantiles = np.linspace(0.01, 0.99, num_branches)
    z_scores = norm.ppf(quantiles)
    up_movements = np.exp(drift + volatility * z_scores * sqrt_dt)
    down_movements = 1 / up_movements

    lattice = np.zeros((num_steps + 1, 2 * num_steps * num_branches + 1))
    mid_index = num_steps * num_branches
    for step in range(1, num_steps + 1):
```

```
        for branch in range(-step * num_branches, step * num_branches + 1):
            up_down_ratio = branch / num_branches
            # Correctly determine the movement factor based on the branch and quantiles
            movement_index = abs(branch) if branch < 0 else branch
            movement_factor = np.prod(up_movements[:movement_index] if branch > 0 else down_movements[:movement_index])
            final_stock_price = stock_price * movement_factor

            payoff = max(0, final_stock_price - strike_price) if option_type == 'C' else max(0, strike_price - final_stock_price)
            lattice[step, mid_index + branch] = payoff

    discount_factor = np.exp(-r * time_to_maturity / num_steps)
    for step in range(num_steps, 0, -1):
        for i in range(-step * num_branches + num_branches, step * num_branches - num_branches + 1):
            expected_payoff = sum(
                lattice[step, mid_index + i + branch]
                for branch in range(-num_branches, num_branches + 1)
            ) / (2 * num_branches + 1)
            lattice[step - 1, mid_index + i] = discount_factor * expected_payoff

    return lattice[0, mid_index]
```

We then run this on every row we've extracted from Yahoo Finance. Due to the comprehensive setup of the aforementioned Testing Framework Setup, implementing models for accuracy analysis is incredibly straight forward.

6.4 General Evaluation

Metric	Value
Mean Absolute Difference (MAD)	7.90
Root Mean Squared Error (RMSE)	7.95
Mean Percent Change Difference (MPCD)	6.91%
Correlation Coefficient	0.806

Table 8: Aggregated Performance Metrics of the Hybrid MOPM Model

The aggregated performance metrics present a detailed view of the model's accuracy:

- **Mean Absolute Difference (MAD):** The MOPM registers an average MAD of 7.90, which is lower than the BOPM's MAD of 8.07 and higher than the BSM's 7.65. This quantifies the average magnitude of errors in the MOPM's predictions without considering their direction, indicating a competitive error size within the context of the compared models.
- **Root Mean Squared Error (RMSE):** At an average of 7.95, the RMSE for the MOPM is lower than the BOPM's 8.09 but higher than the BSM's 7.66. This metric provides insight into the error variance, offering a measure of the model's accuracy by highlighting the square root of the average of squared differences between predicted and actual prices. The RMSE indicates that the MOPM is competitively accurate.
- **Mean Percent Change Difference (MPCD):** The MPCD for the MOPM averages 6.91%, which is higher than both the BOPM's 5.68% and the BSM's 6.66%. This metric reflects a

relatively higher percentage deviation between the MOPM's predictions and the actual market prices, indicating a slightly less consistent predictive accuracy in percentage terms compared to BOPM but comparable to BSM.

- **Correlation Coefficient:** The MOPM's average Correlation Coefficient of 0.806 is superior to both the BOPM's 0.749 and the BSM's 0.716, indicating a stronger linear relationship between the predicted and actual prices. This suggests that the MOPM is generally more effective at capturing the direction of price movements, making it a potentially more reliable tool for market analysis.

These comparisons between the MOPM, BSM, and BOPM models provide an insightful look into how each model performs under varying market conditions, highlighting the strengths and limitations of each in predicting market prices accurately.

6.5 Sector Performance

Sector	RMSE	MPDC	Correlation
Aerospace	4.81	0.070	0.955
Consumer Goods	2.96	0.080	0.931
Energy	3.48	0.089	0.827
Financial	4.06	0.045	0.907
Technology	7.34	0.059	0.933

Table 9: Hybrid MOPM Metrics by Sector

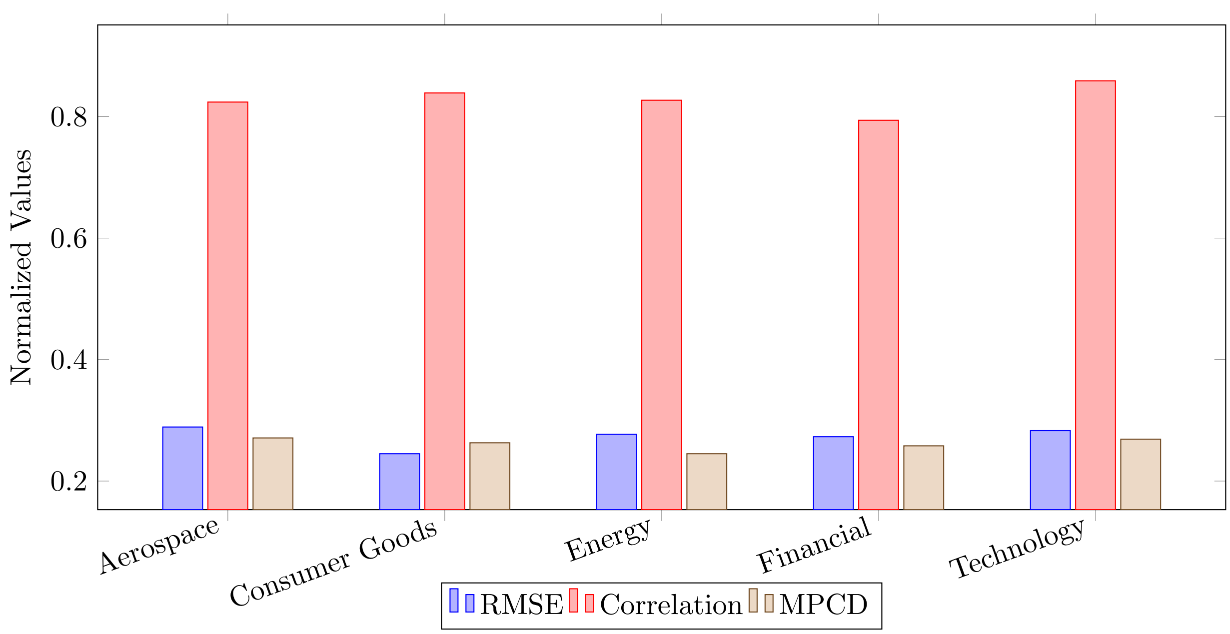


Figure 11: Normalised Hybrid Multinomial Options Pricing Model Metrics by Sector

The evaluation of the Hybrid Mutinomial Options Pricing Model (MOPM) performance across diverse sectors reveals nuanced insights into its efficacy in forecasting market prices amidst sector-specific dynamics, often showing contrast to both the Binomial Options Pricing Model (BOPM) and the Black-Scholes-Merton (BSM) model.

6.5.1 Technology Sector

Apple Inc. (AAPL), *Microsoft Corporation (MSFT)*, and *Nvidia Corporation (NVDA)* etc.

Within the technology sector, the MOPM model shows improved correlation coefficients (0.933) compared to both BOPM (0.677) and BSM (0.662), despite facing high error metrics (RMSE of 7.34). This sector's rapid technological advancements and market disruptions pose unique obstacles for prediction models, suggesting that MOPM, while not reducing error metrics drastically, offers better alignment with market movements.

6.5.2 Aerospace Sector

Boeing Co. (BA) etc.

The aerospace sector demonstrates excellent performance under the MOPM, with the highest correlation coefficient (0.956) compared to both BOPM (0.994) and BSM (0.990). The MOPM shows strong predictability in a sector known for its stability and long-term contracts, suggesting an advantage over BOPM in capturing less volatile market dynamics.

6.5.3 Energy Sector

Exxon Mobil Corp. (XOM) and *Chevron Corp. (CVX)* etc.

The energy sector, while challenging for all models due to fluctuating oil prices and geopolitical tensions, shows modest improvements in the correlation coefficient under MOPM (0.827) compared to BOPM (0.656) and BSM (0.623).

6.5.4 Financial Sector

JPMorgan Chase & Co. (JPM) and *Goldman Sachs Group Inc. (GS)* etc.

The financial sector under MOPM exhibits a higher correlation coefficient (0.907) and compared to BOPM (0.742) and BSM (0.690). This sector's complexity, influenced by economic indicators and policy changes, showcases MOPM's strengths in handling intricate market dynamics more effectively.

6.5.5 Consumer Goods Sector

Procter & Gamble Co. (PG) and *The Coca-Cola Company (KO)* etc.

In the consumer goods sector, MOPM achieves a higher correlation (0.931) compared to BOPM's lower correlation (0.653) and BSM's nearly equal (0.655). This indicates that MOPM is more adept at managing the variability induced by external market forces, enhancing predictability over the other models.

These updated insights provide a detailed comparison between MOPM, BOPM, and BSM, highlighting how sector-specific dynamics can influence the effectiveness of different pricing models in capturing market movements.

6.6 High Volume Asset Performance

High Volume Assets are those that are most highly traded across markets.

The performance of the Modified Options Pricing Model (MOPM) on high-volume assets provides further insights. This section examines assets with significant trading volumes and varying levels of market activity, providing a comparative analysis of the MOPM model's performance metrics.

Performance Metrics Table: The following table provides a detailed breakdown of the MOPM model performance across selected high-volume assets, showcasing the diversity in model accuracy after removing outliers.

Ticker	MOPM RMSE	MOPM Correlation Coefficient	MOPM MPCD (%)
AAPL	4.15	0.815	3.82%
AMD	3.75	0.909	9.17%
AMZN	2.88	0.651	5.56%
META	4.37	0.950	9.13%
MSFT	13.52	0.951	7.61%
NVDA	19.36	0.972	6.46%
TSLA	5.72	0.889	11.16%

Table 10: Performance metrics of the Hybrid MOPM model on selected high volume assets

Differential RMSE and Correlation Coefficients: The RMSE (Root Mean Squared Error) values and correlation coefficients for the MOPM model also show significant variability among high-volume assets. This variability suggests that the MOPM model's efficacy, like that of the BOPM and BSM models, is not uniform across different market conditions and asset classes.

- **Stable Sectors:** Companies such as Amazon (AMZN) and Microsoft (MSFT), which operate in relatively stable sectors like e-commerce and software, demonstrate varied RMSE values with the MOPM model. Amazon shows a lower RMSE and a higher correlation compared to its BSM performance, indicating better prediction accuracy under the MOPM framework. In contrast, Microsoft, with a higher RMSE, reveals ongoing challenges for both models in accurately forecasting stock prices in stable sectors.
- **Volatile Sectors:** Tech companies such as Nvidia (NVDA), Meta Platforms (META), and Tesla (TSLA) exhibit higher RMSE values with the MOPM model, similar to the BSM model. Despite these higher errors, their correlation coefficients are very high (NVDA at 0.972, META at 0.950, and TSLA at 0.889 with MOPM), suggesting that while the MOPM model, like the BSM, captures the direction of price movements well, it struggles with the magnitude, particularly in high-volatility environments such as the technology and automotive sectors. This indicates a consistent limitation across models in dealing with assets subject to rapid market changes and innovations.

7 Evaluations

7.1 Review of Hybrid MOPM Model

Whilst the performance analysis of the Hybird MOPM model put it above the BOPM and BSM model in a few metrics, it is still a long way from an aspirationally useful standalone model. Nevertheless, I believe I have built a highly flexible model that can be adapted to an entire plethora of use cases; this is namely due to the generalised definition of the model - from being able to control the number of branches and time steps, to being able to quickly implement any parameter variations for other types of derivatives (i.e. Compound Options and Options on Non-Equity underlying assets).

My model unfortunately is held back by the same metrics as the BSM and BOPM implementation, where there is a significant discrepancy in the magnitude of simulated options prices - however, it significantly outperforms both with regards to correlation. A side effect of this, is that the model can be used with high confidence in providing trading *signals* - results that aid in providing information about the future **direction** of market movement.

7.2 Objective Completion

We now review the achievement of the objectives I initially set out for this project.

1. Understand historical and current background of option pricing models:

   Through both the discussion of high-level models, as well as the deeper dive into the Black-Scholes-Merton and Binomial Option Pricing models, I feel I have been able to show the knowledge I've gained in regards to the historical progression of this subject.

2. Build knowledge to be able to intuitively understand the derivations of existing models

   The exploration carried out and research presented towards the key steps of derivations for the BSM and BOPM models, give me confidence in having built an intuitive understanding of their inner workings.

3. Research remedies for known pitfalls of existing models

   Similar to the previous point, the presentation of Parameter Improvements, and the implementation of local stochastic volatility into my model which yielded better performance reassures me that the research conducted thoroughly covered a comprehensive understanding into remedies available.

4. Produce a model (or extension of an existing one)

   Whilst it is out of my academic boundaries to have defined a brand-new model, I am more than satisfied with my proposed extension to the existed MOPM model set out by Cheng Few Lee and Jack C. Lee

5. Develop an automated testing framework for models

   A large portion of the successes of this project come from the structure of my testing framework. Once implemented, I did not have to look back and was able to dedicate the rest of my time to understanding and building models.

6. Evaluate existing models and my own one against a range of equities to evaluate performance

> *Once again owing to the effectiveness of my automated testing framework, I was able to evaluate all models holistically across a wide range of equities.*

As such, I believe I was successful in meeting all the criteria that I'd initially set out - and believe I went further, specifically in researching the area of Machine Learning in financial engineering.

7.3 Future Considerations

7.3.1 Integrating ML into the Hybrid Multinomial Approach

The concept

As mentioned earlier, I took some time to look into the possibility of having a Machine Learning algorithm model the up/down probabilities of the lattice structure at every time step. This would theoretically require the model to be trained on a specific option's historical data to refine accuracy. Such ML model would replace the standard normal distribution that I implemented, and under the right training conditions, become more flexible in dealing with the skew of various market parameters.

Existing Work

There are a set of powerful ML models that are often used in financial modelling known as LSTM (Long-Short Term Models) models. These have frequently, and sucessfully, been used for time-series forecasting as a replacement (or alongside) ARIMA (Auto Regressive Integrated Moving Average) models. [27]

The discussed parameter improvements, in conjunction with said LSTM models has shown potential in being a highly effective method of pricing options - as discussed by Que Danfeng [25]

7.3.2 Further parameter improvements

Although having discussed the possibility of improving dividend yield as a parameter into option pricing models, the final implementation of my model did not include it. This is due to it being significantly less binding than volatility.

For models rooted in the foundations of the Black-Scholes model, they are all heavily influenced by volatility as discussed before. As well as this, it is important to not that not all underlying assets pay dividends, and so in many cases any accounting that's been done for dividends would have no effect on the final price modelled.

Nevertheless, if one had the desire to build an all encompassing model, the proposed Hybrid MOPM model could easily be changed to include Stochastic Dividend Yield.

7.3.3 Expansion to non-us markets

We spoke about how we use the 10-year US Treasury to calculate the Risk Free Rate, by assuming it is the hypothetical Risk Free Asset. For other G7 economies, the same applies but you'd use the economy's respective sovereign bond as the risk free asset.

In the scenario where the government is not stable, and there exist no assets in the economy that we can consider as risk-free, there happens to be a handy solution - exchange rates. Take for example we decide to work in ZimDollars modelling an equity based in Zimbabwe with a historically volatile and unsafe economy. We can use the live exchange rate from ZimDollars to a currency backed by some stable economy, such as USD, and then assume the risk free asset is a sovereign bond from the foreign currency, i.e. a US Treasury. Essentially this means we would convert the value of the underlying to USD, and then model an option contract on it against the US Treasury rate and convert that modelled option price back to ZimDollars. We have to convert to USD and back again, as the buy and sell prices of exchange rates (like any asset) aren't the same, and the spread is an important factor to consider.

The reason this works is based on the Fundamental Theorem of Asset Pricing and the efficiency of global markets. In the scenario where we couldn't use a setup as described above, it would lead to the existence of an arbitrage opportunity. As this arbitrage opportunity is eventually spotted and exploited by investors, market forces would cause the opportunity to disappear and allow for the initial setup to hold. This concept is covered under a discussion of Hurdle Rates by Damodaran. [9]

In particular, the model I've defined is well suited for this extension due to the flexibility I've discussed. Similar to how I derived the model to include Local Stochastic Volatility, one can include a simple exchange rate mechanism to not only take exchange rates at face value, but also making it possible to consider further details such a volatility of the exchange rate itself.

7.4 Contribution to public domain

As mentioned at the start of this report, I aimed to add to the information publicly available regarding options pricing. I am hopeful that this document acts as both a source of knowledge for those in the early stages of understanding the theory behind option dynamics, as well as a source of inspiration for those looking for a starting point to deeply investigate derivative market behaviours.

7.5 Ethics, Social Responsibility and Disclaimer

This material is for educational information, and I am not soliciting any action based upon it. This report is not to be construed as an offer to sell or the solicitation of an offer to buy any security in any jurisdiction. This document should in no regards be considered financial advice and the use of the mentioned ideas relay no responsibility to myself. **Certain transactions, including those involving futures, options and high yield securities, give rise to substantial risk and are not suitable for all investors.** Opinions expressed are my present opinions only.

7.6 Project Management

Due to family commitments, illness and unforseen circumstances, there are many points during this year at which I could have fallen behind. As a result of my flexible timeline where I included buffer periods during time sensitive dates (i.e. around coursework deadlines, holidays and exams), I never actually ended up behind where I should have been.

However, I do note that some tasks that I'd set out to complete have been modified as a result of early lack of knowledge as to the depth of the particular field of research. Initially, the project hinted at modelling Compound Options and Greeks, however the two topics are entire dissertations in themselves and at the least require a thorough understanding of the foundations of options.

As such, I pivoted the focus of the project onto said fundamentals - this meant that any work I'd done during the earlier part of the year remained relevant and I had no lost time. See the next page for the proposed project timeline, one that I have consistently kept ahead of.

Another aid during this project was the use of scrum methodologies. Although I was working alone on this project, using a Kanban Board and mock ”sprints” helped me stay accountable to myself whilst maintaining a clear vision of progress I’d been making on the project. It also aided in meetings where I could very easily recall my progress in the project thus far, and what I planned to do in the near future.

References

[1] Waqar Ahmed. Mean absolute error (mae). https://medium.com/@m.waqar.ahmed/understanding-mean-absolute-error-mae-in-regression-a-practical-guide-26e80ebb97df.

[2] Chew Shu Ling Belinda, Chen-Wanhui Sherlyn, Tan-Toh Fei, Edmond C Prakash, and Edmund MK Lai. Visualization of american options using the roll-geske-whaley model. pages 106–1, 2001.

[3] Simon Benninga and Zvi Wiener. The binomial option pricing model. *Mathematica in Education and Research*, 6:27–34, 1997.

[4] Zohra Bi, Abdullah Yousuf, and Mihir Dash. A study on options pricing using garch and black-scholes-merton model. 2014.

[5] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[6] Warren Buffett and Charlie Munger. Discussion on black scholes models. Annual Shareholders Event, 2003. https://www.youtube.com/watch?v=Fd4lfVNJljk.

[7] Chuang-Chang Chang and Hsin-Chang Fu. A binomial option pricing model under stochastic volatility and jump. *Canadian Journal of Administrative Sciences/Revue Canadienne des Sciences de l'Administration*, 18(3):192–203, 2001.

[8] John C Cox, Stephen A Ross, and Mark Rubinstein. Option pricing: A simplified approach. *Journal of Financial Economics*, 7(3):229–263, 1979.

[9] Aswath Damodaran. Estimating hurdle rates - the risk-free rate. https://www.youtube.com/watch?v=584q2f4aTdc, 2014. Video on YouTube.

[10] Emanuel Derman and Iraj Kani. The volatility smile and its implied tree, January 1994. URL https://www.goldmansachs.com/. Quantitative Strategies Research Notes.

[11] Lorella Fatone, Francesca Mariani, Maria Cristina Recchioni, Francesco Zirilli, et al. The barone-adesi whaley formula to price american options revisited. *Applied Mathematics*, 6(2): 382–402, 2015.

[12] Patrick S Hagan, Deep Kumar, Andrew S Lesniewski, and Diana E Woodward. Managing smile risk. *Wilmott Magazine*, 1(12):84–108, 2002.

[13] Espen Gaarder Haug and Nassim Nicholas Taleb. Option traders use (very) sophisticated heuristics, never the black–scholes–merton formula. *Journal of Economic Behavior & Organization*, 77(2):97–106, 2011.

[14] Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.

[15] John C. Hull. *Options, Futures, and Other Derivatives* ***Chapter 1.3, 1.5***. Pearson Education, Upper Saddle River, NJ, 4th edition, 2000. ISBN 0-13-0158224.

[16] John C. Hull. *Options, Futures, and Other Derivatives*. Pearson, 10th edition, 2017.

[17] Investopedia. Correlation coefficient. https://www.investopedia.com/terms/c/correlationcoefficient.asp.

[18] Zuzana Janková. Drawbacks and limitations of black-scholes model for options pricing. *Journal of Financial Studies and Research*, 2018:1–7, 2018.

[19] Cheng Few Lee and Jack C. Lee. *Multinomial Option Pricing Model*, pages 399–408. Springer US, Boston, MA, 2010. ISBN 978-0-387-77117-5. doi: 10.1007/978-0-387-77117-5_25. URL https://doi.org/10.1007/978-0-387-77117-5_25.

[20] Abraham Lioui. Black-scholes-merton revisited under stochastic dividend yields. *Journal of Futures Markets: Futures, Options, and Other Derivative Products*, 26(7):703–732, 2006.

[21] MIT OpenCourseWare. Option price and probability duality. YouTube video, January 6 2015. URL https://www.youtube.com/watch?v=Fd4lfVNJljk. MIT 18.S096 Topics in Mathematics with Applications in Finance, Fall 2013. View the complete course at http://ocw.mit.edu/18-S096F13. Instructor: Stephen Blyth.

[22] Options Clearing Corporation. Annual volume and open interest statistics. https://www.theocc.com, 2024. URL https://www.theocc.com. Data includes options trading volume and open interest statistics from 1973 to 2024.

[23] Hannah Oreskovich. The history of derivatives trading. *MARKET Protocol*, 2018. https://medium.com/market-protocol/the-history-of-derivatives-trading-631e9ab64fed.

[24] Andrea Pascucci. *PDE and Martingale Methods in Option Pricing*. Springer-Verlag, Berlin, 2011.

[25] Danfeng Que. *Option Pricing Using Neural Networks*. PhD thesis, Master's thesis, Science, 2019.

[26] Mark Rubinstein. On the relation between binomial and trinomial option pricing models. 2000.

[27] Sima Siami-Namini, Neda Tavakoli, and Akbar Siami Namin. A comparison of arima and lstm in forecasting time series. In *2018 17th IEEE international conference on machine learning and applications (ICMLA)*, pages 1394–1401. IEEE, 2018.

[28] Kyle Siegrist. Introduction to markov processes, 2022. URL https://stats.libretexts.org/Bookshelves/Probability_Theory/Probability_Mathematical_Statistics_and_Stochastic_Processes_(Siegrist)/16%3A_Markov_Processes/16.01%3A_Introduction_to_Markov_Processes. CC BY 2.0.

[29] Statology. Root mean squared error (rmse). https://www.statology.org/how-to-interpret-rmse/.

[30] Yahoo Finance. Apple inc. (aapl) option derivatives data. https://finance.yahoo.com/quote/AAPL/options?guccounter=1, 2024. URL https://finance.yahoo.com/quote/AAPL/options?guccounter=1. Data on option derivatives for Apple Inc. (AAPL), including strike prices, last price, bid, ask, volume, open interest, and implied volatility for various contract dates and strikes.